Data Structures and Algorithms
University of New Brunswick
Fredericton, New Brunswick, Canada

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Proof Techniques: Proof by Induction

Proof by Induction is a proof technique. In this proof technique, we check that the statement holds for a start value, called *Base case*. Then we assume that the hypothesis is true for $n = k$, called *Induction step*. Finally, using the Induction step, we prove that the statement holds for $n = k + 1$.

Example 0.1. *Prove that* $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

Solution: *As the summation* $\sum_{i=1}^{n} i$ *starts from* $i = 1$*; thus, in the base case we will check whether the statement* $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ *is true for* $n = 1$ *or not. Thus, we write the base case as:*

Base case: *For* $n = 1$, $\sum_{i=1}^{n} i = 1$ *and* $\frac{n(n+1)}{2} = \frac{1(1+1)}{2} = \frac{1\times 2}{2} = 1$. *Thus, the statement* $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ *is true for the base case.*

Induction step: *Now, we assume that the hypothesis is true for* $n = k$. *That is,* $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$ *holds.*

Finally, we need to prove that the hypothesis is true for $n = k + 1$. *That is, we prove that* $\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$.

Let u try to prove it now.

We write as

$$\sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + (k+1). \tag{1}$$

From the Induction step, we have $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$.

Replacing the value of $\sum_{i=1}^{k} i$ *in* (1), *we obtain*

$$\begin{aligned} \sum_{i=1}^{k+1} i &= \frac{k(k+1)}{2} + (k+1) \\ &= \frac{k(k+1) + 2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2} \end{aligned} \tag{2}$$

That is what we wanted to prove. Hence, we conclude the proof.

Example 0.2. *Prove that* $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

Solution: **Base case:** *For* $n = 1$, $\sum_{i=1}^{n} i^2 = 1^2 = 1$ *and* $\frac{n(n+1)(2n+1)}{6} = \frac{1(1+1)(2\times 1+1)}{6} = \frac{1\times 2\times 3}{6} = 1$.

Thus, the statement $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ *is true for the base case,* $n = 1$.

Induction step: *We assume that the hypothesis is true for* $n = k$. *That is,* $\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$ *is true.*

We need to prove that the hypothesis is true for $n = k + 1$. *That is, we prove that* $\sum_{i=1}^{k+1} i^2 = \frac{(k+1)(k+2)(2k+3)}{6}$.

We reformulate as

$$\sum_{i=1}^{k+1} i^2 = \sum_{i=1}^{k} i^2 + (k+1)^2. \tag{3}$$

From the Induction step, we have $\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$.
Replacing the value of $\sum_{i=1}^{k} i^2$ *in* (3), *we obtain*

$$\begin{aligned}\sum_{i=1}^{k+1} i^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\ &= \frac{(k+1)(k(2k+1)+6(k+1))}{6} \\ &= \frac{(k+1)(2k^2+7k+6)}{6} \\ &= \frac{(k+1)(2k^2+3k+4k+6)}{6} \\ &= \frac{(k+1)(k(2k+3)+2(2k+3))}{6} \\ &= \frac{(k+1)(k+2)(2k+3)}{6}\end{aligned} \tag{4}$$

That is what we wanted to prove. Hence, we conclude the proof.

Try these questions:

Question 0.3. *Prove that* $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$.

Question 0.4. *Prove that* $\sum_{i=1}^{n} (2i-1) = n^2$.