(m, n)-Semirings and a Generalized Fault Tolerance Algebra of Systems

A Thesis

Submitted in Partial Fulfillment for the Award of

Master of Technology

In

Information Technology

By

Syed Eqbal Alam

To

International Institute of Information Technology

Bangalore

June 2010

Thesis Certificate

This is to certify that the thesis titled **(m, n)-Semirings and a Generalized Fault Tolerance Algebra of Systems**, submitted by **Syed Eqbal Alam (Roll No. 2008 106)**, to the International Institute of Information Technology, Bangalore, for the award of the degree of **Master of Technology in Information Technology**, is a bona fide record of the research work done by him under my supervision. The contents of this thesis, in full or in parts, have not been submitted to any other Institute or University for the award of any degree or diploma.

Bangalore
The 8^{th} of June, 2010.

Professor Shrisha Rao
Advisor
IIIT-Bangalore

Other Members in the Thesis Committee

Professor K. V. Dinesha

Professor S.S. Prabhu

Dedicated

To

My Parents

Acknowledgements

I would like to express my heartiest gratitude to my advisor Professor Shrisha Rao for his extra ordinary guidance and support. His personal and professional guidance has been of immense source of inspiration during the entire study period at IIIT-B. He has not only extended his expert professional guidance in research but also helped me with his vast reservoir of knowledge and human skill in my personality development. I am highly obliged and indebted to him for his distinguished guidance and moral encouragement, without his guidance this thesis would have been a dream for me.

I would like to thank Professor K.V. Dinesha and Professor S.S. Prabhu for serving on the thesis committee and for generously giving time and precious comments to improve the work.

I am grateful to Professor Bijan Davvaz of Yazd University, Iran for his valuable suggestions and for verifying the document.

I am also thankful to Professor Jayanta Biswas and Dr. Partha Ray who have supervised me in the third semester when I was working on wireless sensor networks.

I would also like to thank my co-researchers Balkrishnan V. and Chandrashekhar Lavania for the brainstorming discussions we have.

I am also obliged to all the faculty members and the staff of IIIT-B for their help during the study period. Thanks are also due to IIIT-B , of which the resources I have utilized.

I am indeed indebted to my parents for their love and affection and my brothers Syed Abrar Alam and Syed Mahfooz Alam and other family members for their moral support and encouragement which has been a source of inspiration and moral boost up.

Last but not the least, I would like to thank all those people who have helped and encouraged me to do this research.

Abstract

We propose a new class of mathematical structures called (m, n)-*semirings* (which generalize the usual semirings), and describe their basic properties. We also define partial ordering, and generalize the concepts of congruence, homomorphism, etc., for (m, n)-semirings. Following earlier work by Rao [37], we consider a system as made up of several components whose failures may cause it to fail, and represent the set of systems algebraically as an (m, n)-semiring. Based on the characteristics of these components we present a formalism to compare the fault tolerance behavior of two systems using our framework of a partially ordered (m, n)-semiring. Assuming **0** as a system which is "always up" and **1** as a system which is "always down", based on these assumptions we prove which system is more fault tolerant. We also compare the fault tolerance behavior of two congruent system using the congruence relation. We check whether two systems are congruent or not based on their fault tolerance behavior. We have also mentioned an example of wireless sensor network and comparison of fault tolerance in two such networks.

Key words:

(m, n)-semiring, components, fault tolerance, partial ordering, congruent system

Contents

1 Introduction **1**

1.1 An Overview of Fault Tolerance 1

1.2 Motivation and Background 2

1.2.1 Semirings and Their Fundamental Properties 2

1.2.2 System Semiring and Fault Tolerance Partial Ordering . . . 5

1.3 Arrangement of Chapters 9

2 (m, n)-Semirings and Their Properties **10**

2.1 Preliminaries 10

2.2 Definition and Examples of (m, n)-Semirings 12

2.3 Identity Elements 14

2.4 Idempotents of (m, n)-Semirings 17

2.5 Congruence Relation 18

2.6 Homomorphism and Isomorphism 19

2.7 (m, n)-Subsemiring 21

3 Partial Ordering On Fault Tolerance **24**

3.1 Definition and Example of f and g 24

3.2 Fault Tolerance Partial Ordering 26

3.3 Partially Ordered (m, n)-Semirings 27
3.4 Main Results 28

4 Fault Tolerance In Congruent Systems **44**

4.1 Background 44
4.2 (m, n)-Semiring Congruence Relation 45

References **58**

Table of Notations 1

Notations	Signifies	Page No.
$\mathcal{R}$	non empty set	2
$\mathbb{Z}$	set of integers	10
$\mathbb{Z}_+$	set of all positive integers	3
$\mathbb{Z}_-$	set of all negative integers	10
$+$	addition operation	3
$\times$	multiplication operation	2
$(\mathcal{R}, \times)$	groupoid	2
$(\mathcal{R}, +, \times)$	semiring	3
$\cup$	union operation	3
$\cap$	intersection operation	3
φ	mapping from one semiring to another semiring	4
ψ	mapping from one semiring to another semiring	5
$\psi \circ \varphi$	product of ψ and φ	5
$\ker \varphi$	kernel of φ	20
$\mathcal{U}$	set of all systems in the domain of discourse	5
x_i	component in $\mathcal{U}$	25
$\preccurlyeq$	partial ordering relation	6
$(\mathcal{U}, \preccurlyeq)$	partially ordered set	6
$(\mathcal{U}, +, \times, \preccurlyeq)$	partially ordered semiring	6
$\mathbf{0}$	additive identity element	6
$\mathbf{1}$	multiplicative identity element	6

Table of Notations 2

Notations	Signifies	Page No.
A	a system in $\mathcal{U}$	6
$\sum$	summation	8
$\prod$	product	8
m	a positive integer	10
n	a positive integer	10
f	m-ary operator	10
g	n-ary operator	12
x_i^m	$x_i, x_{i+1}, \ldots, x_m$	10
$\overset{(m)}{f}(f(x))$	$f(\underbrace{f(x), f(x), \ldots, f(x)}_{m})$	11
$(\mathcal{R}, f, g)$	(m,n)-semiring	12
$\cong$	congruence relation	18
$\mathcal{R}/\cong$	quotient of $\mathcal{R}$ by $\cong$	18
$(\mathcal{R}, f, g, \preccurlyeq)$	partially ordered (m,n)-semiring	27

Chapter 1

Introduction

1.1 An Overview of Fault Tolerance

Fault is defined as a "defect" at the lowest level of abstraction [30]. Thus a "fault" is related to the components of a system due to which the components may fail or give error[36]. Fault tolerance is the property of a system to be functional even if some of its components fail. As stated by Abbott [1] " a system is fault tolerant to the extent that it can prevent faults from causing failure". It is a very critical issue in the design of the systems as in Air Traffic Control Systems [13, 9], real-time embedded systems [6], robotics [22, 33], automation systems [5, 12], medical systems [17], mission critical systems [35] and a lot of other places. In 1967 the paper "design of fault tolerant computers" [4] appeared which presents a notion of fault-tolerant systems and types of faults. Cristian [11] presents basic concepts for designing a fault tolerant distributed systems, Jalote [30] puts forth the fundamental concepts of fault tolerance in distributed systems. We use the fact that each system consists of components or sub-systems, the fault tolerance behavior of the system depends on each of the components or sub-systems that constitute the system. A system may itself be a module or part of a larger system, so that its fault-tolerance affects that of the whole system of which it is a part. In Chapter 3 we analyze the fault tolerance of a system using the proposed new class of algebraic structure (m, n)-semiring extending the earlier work of Rao [37].

Fault tolerance modeling using algebraic structures is proposed by Beckmann [7] for groups, and by Hadjicostis [27] for semigroups and semirings. Semirings are also used in other areas of computer science like cryptography [34], databases [26], graph theory, game theory [25], etc. Rao [37] uses the formalism of semirings to analyze the fault-tolerance of a system as a function of its composition, with a partial ordering relation between systems used to compare their fault-tolerance behaviors.

1.2 Motivation and Background

In this section we define semiring and mention some of the basic properties of it and we also define system semiring and its properties as defined by Rao [37]. Subsection 1.2.1 deals with the definition and fundamental properties of semiring which we have generalized in Chapter 2 and Subsection 1.2.2 deals with the properties of system semiring and fault tolerance partial ordering which we have generalized in Chapter 3.

1.2.1 Semirings and Their Fundamental Properties

Definition 1.2.1. Let $\mathcal{R}$ be a non empty set.

(i) $\mathcal{R}$ with a binary operation "$\times$" is called a *groupoid* and is denoted by $(\mathcal{R}, \times)$ [29].

(ii) If elements of $(\mathcal{R}, \times)$ also follow the associative law then $(\mathcal{R}, \times)$ is called a *semigroup*, i.e., if elements x, y, z be in $\mathcal{R}$ and $x \times (y \times z) = (x \times y) \times z$ then $(\mathcal{R}, \times)$ is a semigroup [29].

(iii) The algebraic structure $(\mathcal{R}, +, \times)$ which supports the addition $(+)$ and multiplication $(\times)$ operations, and satisfies the following conditions is called a *semiring*:

  (a) $(\mathcal{R}, +)$ is a commutative semigroup,

  (b) $(\mathcal{R}, \times)$ is a semigroup, and

  (c) Multiplication distributes over addition [25, 28].

Example 1.2.2. Let $\mathbb{Z}_+$ be a set of all positive integers, then $(\mathbb{Z}_+, +, \times)$ is a semiring, where $+$ and $\times$ has its usual meaning of addition and multiplication operations [29].

Example 1.2.3. Let $\mathcal{B}$ be a set. Then $(\mathcal{B}, \cup, \cap)$ is a semiring, where $\cup$ denotes union and $\cap$ denotes the intersection operations of two elements [29].

Definition 1.2.4. We restate following properties of a semiring as defined by Hebisch and Weinert [28]:

(i) The semigroup $(\mathcal{R}, +)$ has an *identity element* $\mathbf{0}$, then if $x \in \mathcal{R}$, the following relation holds $(x + \mathbf{0}) = x$. Similarly the semigroup $(\mathcal{R}, \times)$ has an *identity element* $\mathbf{1}$, then if $y \in \mathcal{R}$, the following relation holds $(y \times \mathbf{1}) = y$.

(ii) We define *multiplicatively absorbing* $\mathbf{0}$ if $x \times \mathbf{0} = \mathbf{0} = \mathbf{0} \times x$ for all $x \in \mathcal{R}$.

(iii) x is called an *idempotent* of semigroup $(\mathcal{R}, +)$ if $x+x = x, \forall x \in \mathcal{R}$. A semiring $(\mathcal{R}, +, \times)$ is called *additively idempotent*, if $(\mathcal{R}, +)$ is an idempotent semigroup, similarly we define *multiplicatively idempotent*, if $(\mathcal{R}, \times)$ is an idempotent, i.e, $y \times y = y$ for all $y \in \mathcal{R}$.

We mention some of the results of Chapter I of Hebisch and Weinert [28] as follows:

Theorem 1.2.5. A semiring $(\mathcal{R}, +, \times)$ with an identity element $\mathbf{0}$ is additively idempotent if and only if $\mathbf{0} + \mathbf{0} = \mathbf{0}$ holds.

Theorem 1.2.6. A semiring $(\mathcal{R}, +, \times)$ having at least two additively idempotent elements is not additively cancellative.

Proof. Let a, b be two additively idempotent elements in $\mathcal{R}$ and $a \neq b$.

As $a + b = a + b$, using idempotent property of elements a, b, we can write it as follows:

$$a + a + b = a + b + b$$

which can be written as

$$a + (a + b) = (a + b) + b$$

let the semiring $(\mathcal{R}, +, \times)$ is additively cancellative, then

$$a + (a + b) = (a + b) + b \Longrightarrow a = b$$

which is a contradiction to our assumption that $a \neq b$, thus $(\mathcal{R}, +, \times)$ is not additively cancellative. □

Definition 1.2.7. Let $(\mathcal{R}, +, \times)$ and $(\mathcal{S}, +, \times)$ be semirings. Then the mapping $\varphi : \mathcal{R} \to \mathcal{S}$ is called *homomorphism* of $(\mathcal{R}, +, \times)$ into $(\mathcal{S}, +, \times)$ if following hold

(i) $\varphi(x + y) = \varphi(x) + \varphi(y)$ and

(ii) $\varphi(x \times y) = \varphi(x) \times \varphi(y)$ for all $x, y \in \mathcal{R}$.

Definition 1.2.7 is Definition 3.1 in [28].

Theorem 1.2.8. Let $(\mathcal{R}, +, \times)$, $(\mathcal{S}, +, \times)$ and $(\mathcal{T}, +, \times)$ be semirings. Then if the following mappings $\varphi : (\mathcal{R}, +, \times) \to (\mathcal{S}, +, \times)$ and $\psi : (\mathcal{S}, +, \times) \to (\mathcal{T}, +, \times)$ are homomorphisms, then

$\psi \circ \varphi : (\mathcal{R}, +, \times) \to (\mathcal{T}, +, \times)$ is also a homomorphism.

For more details about the properties of semirings we can check [25, 28, 23].

1.2.2 System Semiring and Fault Tolerance Partial Ordering

In this section we use $\mathcal{U}$ which denotes the universal set of all systems in the domain of discourse as given by Rao [37]. We use *component* to refer to an *atomic* part of a system, i.e., which has no component or sub-system of its own, and *subsystem* to refer to a part of a system that is not necessarily atomic. We assume that components and subsystems are *disjoint*, in the sense that if fail, they fail independently and do not affect the functioning of other components [37].

Definition 1.2.9. As stated by Rao [37], we define the direct sum $+$ and direct product $\times$ as following:

(i) The $+$ operator is considered to apply for systems consisting of two components when the failure of either would cause the system as a whole to fail.

(ii) The $\times$ operator applies for systems consisting of two components when the failure of both is necessary to cause the system as a whole to fail.

Remark 1.2.10. Let a, b are disjoint components, which are in $\mathcal{U}$ then the system constituted by applying $+$ operator is $a+b$, this system fails when either of a and b fail. And the system constituted by applying $\times$ operator is $a \times b$, this system fails when both the components fail [37].

Definition 1.2.11. (i) We denote *system semiring* for the binary operation $+$ and $\times$ as $(\mathcal{U}, +, \times)$, which satisfies all conditions of Definition 1.2.1 (iii) [37].

(ii) Identity elements of a semiring are defined as follows [37]:

(a) The additive identity $\mathbf{0}$ is the system such that for any system A, $A + \mathbf{0} = \mathbf{0} + A = A$.

(b) The multiplicative identity $\mathbf{1}$ is the system such that for any system A, $A \times \mathbf{1} = \mathbf{1} \times A = A$.

(iii) If $\preccurlyeq$ be a partial ordering relation on $\mathcal{U}$, such that $(\mathcal{U}, \preccurlyeq)$ is a poset then this is called as *fault-tolerance partial ordering* where $A \preccurlyeq B$ means that A has a lower measure of some fault metric than B (e.g., A has fewer failures per hour than B, or has a better fault tolerance than B) for all $A, B \in \mathcal{U}$ [37].

(iv) $(\mathcal{U}, +, \times, \preccurlyeq)$ is a *partially ordered semiring* if the following conditions are satisfied, for all A, B, and C in $\mathcal{U}$ [28, 37].

(a) The *monotony law of addition*:

$$A \preccurlyeq B \longrightarrow A + C \preccurlyeq B + C$$

(b) The *monotony law of multiplication*:

$$A \preccurlyeq B \longrightarrow A \times C \preccurlyeq B \times C.$$

Remark 1.2.12. We assume $\mathbf{0}$ is a system which is "always up" and $\mathbf{1}$ is a system which is "always down" then for all $x \in \mathcal{U}$, $\mathbf{0} \preccurlyeq x$ which states that $\mathbf{0}$ is more fault tolerant than x. Similarly, $y \preccurlyeq \mathbf{1}$ for all $y \in \mathcal{U}$ states that y is more fault tolerant than $\mathbf{1}$ [37].

We mention some of the results of Rao [37] below. For the proof of these results please refer to [37].

Lemma 1.2.13. If $\preccurlyeq$ is a fault-tolerance partial order, then $\forall A, B \in \mathcal{U}$:

(i) $A \preccurlyeq A + B$, and

(ii) $A \times B \preccurlyeq A$.

This states that, the system constituted by applying + operator on on its components or subsystems is less fault tolerant than a single component or subsystem. Similarly if we apply $\times$ on two components or subsystems then any subsystem or component is less fault tolerant than the constituted system.

Lemma 1.2.14. If $\preccurlyeq$ is a fault-tolerance partial order, then $\forall n \in \mathbb{Z}_+$ and $\forall A \in \mathcal{U}$,

(i) $A \preccurlyeq nA$, and

(ii) $A^n \preccurlyeq A$.

Theorem 1.2.15. If $A \preccurlyeq B$ and $X \preccurlyeq Y$, where $\preccurlyeq$ is a partial order, then following hold:

(i) $A + X \preccurlyeq B + Y$ and

(ii) $A \times X \preccurlyeq B \times Y$

This theorem states that if one component or subsystem of a system is less fault tolerant than the other subsystem of some other system, similarly the other component also then the system constituted is less fault tolerant than the other system.

Corollary 1.2.16. If $\preccurlyeq$ is a fault-tolerance partial order and if $A \preccurlyeq B$, then $\forall n \in \mathbb{Z}_+$,

(i) $nA \preccurlyeq nB$, and

(ii) $A^n \preccurlyeq B^n$.

Corollary 1.2.17. The following hold for all $A, B \in \mathcal{U}$ and all $n \in \mathbb{Z}_+$:

(i) if $nA \preccurlyeq B$, then $A \preccurlyeq B$; and

(ii) if $A \preccurlyeq B^n$, then $A \preccurlyeq B$.

Theorem 1.2.18. If $A_i \preccurlyeq B_i$, with $0 \leq i \leq n-1$ and $A_i, B_i \in \mathcal{U}$, then:

$$\sum_{i=0}^{n-1} A_i \preccurlyeq \sum_{i=0}^{n-1} B_i,$$

and

$$\prod_{i=0}^{n-1} A_i \preccurlyeq \prod_{i=0}^{n-1} B_i.$$

In this Thesis, we first define the (m, n)-semiring $(\mathcal{R}, f, g)$, (which is a generalization of the ordinary semiring $(\mathcal{R}, +, \times)$, where $\mathcal{R}$ is a set with binary operations $+$ and $\times$), using f and g which are m-ary and n-ary operations respectively. We propose identity elements, multiplicatively absorbing elements, idempotents, homomorphism, subsemiring, center and i-center of the (m, n)-semiring. We also briefly touch on zero-divisor free, zero-sum free, additively cancellative, and multiplicatively cancellative (m, n)-semirings, and the congruence relation on (m, n)-semirings.

1.3 Arrangement of Chapters

In Chapter 2 we use $\mathcal{R}$ to represent the set, whereas in Chapter 3 and Chapter 4, $\mathcal{U}$ is used to represent the set of all systems and talk about system (m, n)-semiring.

Chapter 2 deals with the notations used, the general conventions followed and definition and basic properties of (m, n)-semiring. In Chapter 3 we extend the results of Rao [37] using partially ordered (m, n)-semiring. A class of systems is algebraically represented by an (m, n)-semiring, and the fault tolerance behavior of two systems is compared using partially ordered (m, n)-semiring.

Chapter 4 explains about the congruence relation and fault tolerance behavior of congruent systems. In this Chapter we compare the fault tolerance behavior of two systems.

Chapter 2

(m, n)-Semirings and Their Properties

2.1 Preliminaries

The set of integers is denoted by $\mathbb{Z}$, with $\mathbb{Z}_+$ and $\mathbb{Z}_-$ denoting the sets of positive integers and negative integers respectively. Let $\mathcal{R}$ be a set and f be a mapping $f : \mathcal{R}^m \rightarrow \mathcal{R}$, i.e., f is an m-ary operator, likewise g is an n-ary operator, where $m, n \in \mathbb{Z}_+$. Elements of the set $\mathcal{R}$ are denoted by x_i, y_i where $i \in \mathbb{Z}_+$.

In Chapter 3, $\preccurlyeq$ denotes fault-tolerance partial order, $\mathcal{U}$ is used to denote the set of all systems in the domain of discourse, letters with subscript like a_i denote the components which are in $\mathcal{U}$, where $i \in \mathbb{Z}_+$.

In Chapter 4, $\cong$ denotes the congruence relation.

We use following general convention as followed by [21, 20, 14]:

The sequence x_i, x_{i+1}, $\ldots$, x_m is denoted by x_i^m.

The following term:

$$f(x_1, \dots, x_i, y_{i+1}, \dots, y_j, z_{j+1}, \dots, z_m) \tag{2.1}$$

is represented as:

$$f(x_1^i, y_{i+1}^j, z_{j+1}^m) \tag{2.2}$$

In the case when $y_{i+1} = \dots = y_j = y$, then (2.2) is expressed as:

$$f(x_1^i, \overset{(j-i)}{y}, z_{j+1}^m)$$

If $x_1 = \dots = x_i = y_{i+1} = \dots = y_j = z_{j+1} = \dots = z_m = f(a_1^m)$, then (2.1) is represented as:

$$f(\overset{(m)}{f(a_1^m)})$$

Definition 2.1.1. (i) A nonempty set $\mathcal{R}$ with an m-ary operation f is called an *m-ary groupoid* and is denoted by $(\mathcal{R}, f)$ (see Dudek [20]).

(ii) Let $x_1, x_2, \dots, x_{2m-1} \in \mathcal{R}$. Then the associativity and distributivity laws for the m-ary operator f are defined as follows:

(a) *Associativity*:

$f(x_1^{i-1}, f(x_i^{m+i-1}), x_{m+i}^{2m-1}) = f(x_1^{j-1}, f(x_j^{m+j-1}), x_{m+j}^{2m-1})$

for all $x_1, \dots, x_{2m-1} \in \mathcal{R}$ and holds for all $1 \leq i \leq j \leq m$ (from Gluskin [24]).

(b) *Commutativity*:

$f(x_1, x_2, \dots, x_m) = f(x_{\eta(1)}, x_{\eta(2)}, \dots, x_{\eta(m)})$

for every permutation η of $\{1, 2, \dots, m\}$ (from Timm [39]), $\forall x_1, x_2, \dots, x_m \in$

$\mathcal{R}$.

Remark 2.1.2. For all $x, y, a \in \mathcal{R}$, the following is commutative (from Dudek [18]):

$$f(x, \underbrace{a, \ldots, a}_{m-2}, y) = f(y, \underbrace{a, \ldots, a}_{m-2}, x).$$

Definition 2.1.3. Let $\mathcal{R}$ be a set.

(i) An m-ary groupoid $(\mathcal{R}, f)$ is called an *m-ary semigroup* if f is associative (from Dudek [20]) i.e, if

$$f(x_1^{i-1}, f(x_i^{m+i-1}), x_{m+i}^{2m-1}) = f(x_1^{j-1}, f(x_j^{m+j-1}), x_{m+j}^{2m-1})$$

for all $x_1, \ldots, x_{2m-1} \in \mathcal{R}$, where $1 \leq i \leq j \leq m$.

(ii) Let $x_1, x_2, \ldots, x_n, a_1, a_2, \ldots, a_m$ be elements of set $\mathcal{R}$ and $1 \leq i \leq n$. The n-ary operator g is *distributive* with respect to the m-ary operator f if:

$$g(x_1^{i-1}, f(a_1^m), x_{i+1}^n) = f(g(x_1^{i-1}, a_1, x_{i+1}^n), \ldots, g(x_1^{i-1}, a_m, x_{i+1}^n)).$$

Remark 2.1.4. Consider a k-ary group (G, h) in which the k-ary operation h is distributive with respect to itself, i.e.,

$$h(x_1^{i-1}, h(a_1^k), x_{i+1}^k) = h(h(x_1^{i-1}, a_1, x_{i+1}^k), \ldots, h(x_1^{i-1}, a_k, x_{i+1}^k)),$$

for all $1 \leq i \leq k$. These type of groups are called *autodistributive* k-ary groups (Dudek [19]).

2.2 Definition and Examples of (m, n)-Semirings

Definition 2.2.1. An (m, n)-semiring is an algebraic structure $(\mathcal{R}, f, g)$ which satisfies the following axioms:

(i) $(\mathcal{R}, f)$ is an m-ary semigroup,

(ii) $(\mathcal{R}, g)$ is an n-ary semigroup,

(iii) the n-ary operator g is *distributive* with respect to the m-ary operation f, i.e., for every $x_1, x_2, \ldots, x_n, a_1, a_2, \ldots, a_m \in \mathcal{R}, 1 \leq i \leq n$,

$$g(x_1^{i-1}, f(a_1^m), x_{i+1}^n) = f(g(x_1^{i-1}, a_1, x_{i+1}^n), \ldots, g(x_1^{i-1}, a_m, x_{i+1}^n))$$

Example 2.2.2. Let $\mathcal{B}$ be any Boolean algebra. Then $(\mathcal{B}, f, g)$ is an (m, n)-semiring where $f(A_1^m) = A_1 \cup A_2 \cup \ldots \cup A_m$ and $g(B_1^n) = B_1 \cap B_2 \cap \ldots \cap B_n$, for all $A_1, A_2, \ldots, A_m$ and $B_1, B_2, \ldots, B_n \in \mathcal{B}$.

In general, we have the following

Theorem 2.2.3. Let $(\mathcal{R}, +, \times)$ be an ordinary semiring. Let f be an m-ary operation and g be an n-ary operation on $\mathcal{R}$ as follows:

$$f(x_1^m) = \sum_{i=1}^{m} x_i, \qquad \forall x_1, x_2, \ldots, x_m \in \mathcal{R},$$

$$g(y_1^n) = \prod_{i=1}^{n} y_i, \qquad \forall y_1, y_2, \ldots, y_n \in \mathcal{R}.$$

Then $(\mathcal{R}, f, g)$ is an (m, n)-semiring.

Proof. Omitted as obvious. □

Example 2.2.4. The following give us some (m, n)-semirings in different ways indicated by Theorem 2.2.3.

(i) Let $(\mathcal{R}, +, \times)$ be an ordinary semiring and $x_1, x_2, \ldots, x_n$ be in $\mathcal{R}$. If we set:

$$g(x_1^n) = x_1 \times x_2 \times \ldots \times x_n.$$

Then we get a $(2,n)$-semiring $(\mathcal{R},+,g)$.

(ii) The set $\mathbb{Z}_-$ of all negative integers is not closed under the binary products, i.e., $\mathbb{Z}_-$ does not form a semiring, but it is a $(2,3)$-semiring.

2.3 Identity Elements

Definition 2.3.1. Let $(\mathcal{R},f,g)$ be an (m,n)-semiring. Then m-ary semigroup $(\mathcal{R},f)$ has an *identity element* $\mathbf{0}$ such that

$$x = f(\underbrace{\mathbf{0},\ldots,\mathbf{0}}_{i-1},x,\underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-i})$$

for all $x \in \mathcal{R}$ and $1 \leq i \leq m$, We call $\mathbf{0}$ as an *identity element* of (m,n)-semiring $(\mathcal{R},f,g)$.

Similarly, n-ary semigroup $(\mathcal{R},g)$ has an *identity element* $\mathbf{1}$ such that

$$y = g(\underbrace{\mathbf{1},\ldots,\mathbf{1}}_{j-1},y,\underbrace{\mathbf{1},\ldots,\mathbf{1}}_{n-j})$$

for all $y \in \mathcal{R}$ and $1 \leq j \leq n$.

We call $\mathbf{1}$ as an *identity element* of (m,n)-semiring $(\mathcal{R},f,g)$.

We therefore call $\mathbf{0}$ the f-identity, and $\mathbf{1}$ the g-identity.

Remark 2.3.2. In an (m,n)-semiring $(\mathcal{R},f,g)$, placing $\mathbf{0}$ and $\mathbf{1}$, $(m-2)$ and $(n-2)$ times respectively, we obtain the following binary operations:

$$x+y = f(x,\underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-2},y) \text{ and } x\times y = g(x,\underbrace{\mathbf{1},\ldots,\mathbf{1}}_{n-2},y), \qquad \text{for all } x,y\in\mathcal{R}.$$

Definition 2.3.3. Let $(\mathcal{R},f,g)$ be an (m,n)-semiring with an f-identity element $\mathbf{0}$ and g-identity element $\mathbf{1}$. Then:

(i) $\mathbf{0}$ is said to be *multiplicatively absorbing* if it is absorbing in $(\mathcal{R}, g)$, i.e., if

$$g(\mathbf{0}, x_1^{n-1}) = g(x_1^{n-1}, \mathbf{0}) = \mathbf{0}$$

for all $x_1, x_2, \ldots, x_{n-1} \in \mathcal{R}$.

(ii) $(\mathcal{R}, f, g)$ is called *zero-divisor free* if

$$g(x_1, x_2, \ldots, x_n) = \mathbf{0}$$

always implies $x_1 = \mathbf{0}$ or $x_2 = \mathbf{0}$ or $\ldots$ or $x_n = \mathbf{0}$.

Elements $x_1, x_2, \ldots, x_{n-1} \in \mathcal{R}$ are called *left zero-divisors* of (m, n)-semiring $(\mathcal{R}, f, g)$ if there exists $a \neq \mathbf{0}$ and the following holds:

$$g(x_1^{n-1}, a) = \mathbf{0}.$$

(iii) $(\mathcal{R}, f, g)$ is called *zero-sum free* if

$$f(x_1, x_2, \ldots, x_m) = \mathbf{0}$$

always implies $x_1 = x_2 = \ldots = x_m = \mathbf{0}$.

(iv) $(\mathcal{R}, f, g)$ is called *additively cancellative* if the m-ary semigroup $(\mathcal{R}, f)$ is cancellative, i.e.,

$$f(x_1^{i-1}, a, x_{i+1}^m) = f(x_1^{i-1}, b, x_{i+1}^m) \quad \Longrightarrow a = b$$

for all $a, b, x_1, x_2, \ldots, x_m \in \mathcal{R}$.

(v) $(\mathcal{R}, f, g)$ is called *multiplicatively cancellative* if the n-ary semigroup $(\mathcal{R}, g)$ is cancellative, i.e.,

$$g(x_1^{i-1}, a, x_{i+1}^n) = g(x_1^{i-1}, b, x_{i+1}^n) \quad \Longrightarrow a = b$$

for all $a, b, x_1, x_2, \ldots, x_n \in \mathcal{R}$.

Elements $x_1, x_2, \ldots, x_{n-1}$ are called *left cancellable* in an n-ary semigroup $(\mathcal{R}, g)$ if

$$g(x_1^{n-1}, a) = g(x_1^{n-1}, b) \quad \Longrightarrow a = b$$

for all $x_1, x_2, \ldots, x_{n-1}, a, b \in \mathcal{R}$.

$(\mathcal{R}, f, g)$ is called *multiplicatively left cancellative* if elements $x_1, x_2, \ldots, x_{n-1} \in \mathcal{R} \setminus \{\mathbf{0}\}$ are multiplicatively left cancellable in n-ary semigroup $(\mathcal{R}, g)$.

Theorem 2.3.4. Let $(\mathcal{R}, f, g)$ be an (m, n)-semiring with f-identity $\mathbf{0}$.

(i) If elements $x_1, x_2, \ldots, x_{n-1} \in \mathcal{R}$ are multiplicatively left cancellable, then elements $x_1, x_2, \ldots, x_{n-1}$ are not left divisors.

(ii) If the (m, n)-semiring $(\mathcal{R}, f, g)$ is multiplicatively left cncellative, then it is zero-divisor free.

We have generalized Theorem 2.3.4 from Theorem 4.4 of Hebisch and Weinert [28].

We have generalized the definition of idempotents of semirings given by Bourne [8] and Hebisch and Weinert [28]), as following.

2.4 Idempotents of (m, n)-Semirings

Definition 2.4.1. Let $(\mathcal{R}, f, g)$ be an (m, n)-semiring. Then:

(i) It is called *additively idempotent* if $(\mathcal{R}, f)$ is an idempotent m-ary semigroup, i.e., if

$$f(\underbrace{x, x, \ldots, x}_{m}) = x$$

for all $x \in \mathcal{R}$.

(ii) It is called *multiplicatively idempotent* if $(\mathcal{R}, g)$ is an idempotent n-ary semigroup, i.e., if

$$g(\underbrace{y, y, \ldots, y}_{n}) = y$$

for all $y \in \mathcal{R}$, $y \neq \mathbf{0}$.

Theorem 2.4.2. An (m, n)-semiring $(\mathcal{R}, f, g)$ having at least two multiplicatively idempotent elements is not multiplicatively cancellative.

Proof. Let a and b be two multiplicatively idempotent elements, $a \neq b$. Then:

$$g(\underbrace{\mathbf{1}, \ldots, \mathbf{1}}_{n-2}, g(\overset{(n)}{a}), b) = g(\underbrace{\mathbf{1}, \ldots, \mathbf{1}}_{n-2}, a, g(\overset{(n)}{b}))$$

which is represented as:

$$g(\underbrace{\mathbf{1}, \ldots, \mathbf{1}}_{n-3}, g(\mathbf{1}, \overset{(n-1)}{a}), a, b) = g(\underbrace{\mathbf{1}, \ldots, \mathbf{1}}_{n-3}, a, b, g(\mathbf{1}, \overset{(n-1)}{b})).$$

If the (m, n)-semiring $(\mathcal{R}, f, g)$ is multiplicatively cancellative, then the following holds true:

$$g(\underbrace{\mathbf{1}, \ldots, \mathbf{1}}_{n-1}, g(\mathbf{1}, \overset{(n-1)}{a})) = g(\underbrace{\mathbf{1}, \ldots, \mathbf{1}}_{n-1}, g(\mathbf{1}, \overset{(n-1)}{b})),$$

$$g(\mathbf{1}, \overset{(n-1)}{a}) = g(\mathbf{1}, \overset{(n-1)}{b}),$$

which implies that $a = b$, which is a contradiction to the assumption that $a \neq b$, so that $(\mathcal{R}, f, g)$ is not multiplicatively cancellative. □

We have generalized Exercise 2.7 in Chapter I of Hebisch and Weinert [28] to get the following.

Theorem 2.4.3. An (m, n)-semiring (R, f, g) having an f-identity element $\mathbf{0}$ is additively idempotent if and only if the following holds:

$$f(\underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m}) = \mathbf{0}.$$

2.5 Congruence Relation

Definition 2.5.1. Let $(\mathcal{R}, f, g)$ be an (m, n)-semiring and $\cong$ be an equivalence relation on $\mathcal{R}$.

(i) Then $\cong$ is called a *congruence relation* or a *congruence* of $(\mathcal{R}, f, g)$, if it satisfies the following properties:

  (a) if $x_i \cong y_i$ then $f(x_1^m) \cong f(y_1^m)$ for all $1 \leq i \leq m$; and,

  (b) if $z_j \cong u_j$ then $g(z_1^n) \cong g(u_1^n)$ for all $1 \leq j \leq n$,

  for all $x_1, x_2, \ldots, x_m,\ y_1, y_2, \ldots, y_m\ ,z_1, z_2, \ldots, z_n,\ u_1, u_2, \ldots, u_n \in \mathcal{R}$.

(ii) Let $\cong$ be a congruence on an algebra $\mathcal{R}$. Then the *quotient* of $\mathcal{R}$ by $\cong$, written as $\mathcal{R}/\cong$, is the algebra whose universe is $\mathcal{R}/\cong$ and whose fundamental operation satisfy

$f^{\mathcal{R}/\cong}(x_1, x_2, \ldots, x_m) = f^{\mathcal{R}}(x_1, x_2, \ldots, x_m)/\cong$

where $x_1, x_2, \ldots, x_m \in \mathcal{R}$ [10].

Theorem 2.5.2. Let $(\mathcal{R}, f, g)$ be an (m, n)-semiring and the relation $\cong$ be a congruence relation on $(\mathcal{R}, f, g)$. Then the quotient $(\mathcal{R}/\cong, F, G)$ is an (m, n)-semiring under $F((x_1)/\cong, \ldots, (x_m)/\cong) = f(x_1^m)/\cong$ and $G((y_1)/\cong, \ldots, (y_n)/\cong) = g(y_1^n)/\cong$, for all $x_1, x_2, \ldots, x_m$ and $y_1, y_2, \ldots, y_n$ in $\mathcal{R}$.

Proof. Omitted as obvious. □

2.6 Homomorphism and Isomorphism

Definition 2.6.1. We define homomorphism, isomorphism, and a product of two mappings as follows:

(i) A mapping $\varphi : \mathcal{R} \to \mathcal{S}$ from (m, n)-semiring $(\mathcal{R}, f, g)$ into (m, n)-semiring $(\mathcal{S}, f', g')$ is called a *homomorphism* if

$$\varphi(f(x_1^m)) = f'(\varphi(x_1), \varphi(x_2), \ldots, \varphi(x_m))$$

and

$$\varphi(g(y_1^n)) = g'(\varphi(y_1), \varphi(y_2), \ldots, \varphi(y_n))$$

for all $x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n \in \mathcal{R}$.

(ii) The (m, n)-semirings $(\mathcal{R}, f, g)$ and $(\mathcal{S}, f', g')$ are called *isomorphic* if there exists one-to-one homomorphism from $\mathcal{R}$ onto $\mathcal{S}$. One-to-one homomorphism is called *isomorphism*.

(iii) If we apply mapping $\varphi : \mathcal{R} \to \mathcal{S}$ and then $\psi : \mathcal{S} \to \mathcal{T}$ on x we get the mapping $(\psi \circ \varphi)(x)$ which is equal to $\psi(\varphi(x))$, where $x \in \mathcal{R}$. It is called the *product* of

ψ and φ [28].

We have generalized Definition 2.6.1 from Definition 2 of Allen [2].

We have generalized the following theorem from Theorem 3.3 given by Hebisch and Weinert [28].

Theorem 2.6.2. Let $(\mathcal{R}, f, g)$, $(\mathcal{S}, f', g')$ and $(\mathcal{T}, f'', g'')$ be (m, n)-semirings. Then if the following mappings $\varphi : (\mathcal{R}, f, g) \to (\mathcal{S}, f', g')$ and

$\psi : (\mathcal{S}, f', g') \to (\mathcal{T}, f'', g'')$ are homomorphisms, then

$\psi o \varphi : (\mathcal{R}, f, g) \to (\mathcal{T}, f'', g'')$ is also a homomorphism.

Proof. Let $x_1, x_2, \ldots, x_m$ and $y_1, y_2, \ldots, y_n$ be in $\mathcal{R}$. Then:

$$\begin{aligned}
(\psi \circ \varphi)(f(x_1^m)) &= \psi(\varphi(f(x_1, x_2, \ldots, x_m))) \\
&= \psi(f'(\varphi(x_1), \varphi(x_2), \ldots, \varphi(x_m))) \\
&= f''(\psi(\varphi(x_1)), \psi(\varphi(x_2)), \ldots, \psi(\varphi(x_m))) \\
&= f''((\psi \circ \varphi)(x_1), (\psi \circ \varphi)(x_2), \ldots, (\psi \circ \varphi)(x_m)).
\end{aligned}$$

In a similar manner, we can deduce that

$$(\psi \circ \varphi)(g(y_1^n)) = g''((\psi \circ \varphi)(y_1), (\psi \circ \varphi)(y_2), \ldots, (\psi \circ \varphi)(y_n)).$$

Thus it is evident that $\psi \circ \varphi$ is a homomorphism from $\mathcal{R} \to \mathcal{T}$. □

This proof is similar to that of Theorem 6.5 given by Burris and Sankappanavar [10].

Definition 2.6.3. Let $(\mathcal{R}, f, g)$ and $(\mathcal{S}, f', g')$ be (m, n)-semirings, and let $\varphi : \mathcal{R} \to \mathcal{S}$ be a homomorphism. Then the *kernel* of φ, written as $\ker \varphi$ is, following Burris and Sankappanavar [10], as follows:

$$\ker\varphi = \{(a,b)\in\mathcal{R}\times\mathcal{R} \mid \varphi(a)=\varphi(b)\}.$$

2.7 (m,n)-Subsemiring

Definition 2.7.1. Let $\mathcal{S}$ be a non-empty subset of $\mathcal{R}$, where $(\mathcal{R},f,g)$ is an (m,n)-semiring. If $(\mathcal{S},f,g)$ is an (m,n)-semiring, then $\mathcal{S}$ is called an (m,n)-*subsemiring* of $\mathcal{R}$.

Let $(\mathcal{R},f,g)$ be an (m,n)-semiring. By an i-center of $\mathcal{R}$ we mean the set

$$Z_i(\mathcal{R}) = \{a\in\mathcal{R} \mid f(a,x_2^m)=f(x_2^i,a,x_{i+1}^m), \qquad \forall x_2^m\in\mathcal{R}\}.$$

The set $Z(\mathcal{R}) = \bigcap_{i=1}^{m} Z_i(\mathcal{R})$ is called the *center* of $\mathcal{R}$. If $Z_i(\mathcal{R})$ is non-empty, then it is an (m,n)-subsemiring of $\mathcal{R}$. If $Z(\mathcal{R})$ is non-empty, then it is a maximal commutative (m,n)-subsemiring of $\mathcal{R}$.

Lemma 2.7.2. Let $x_1,x_2,\ldots,x_m,y_1,y_2,\ldots,y_n\in\mathcal{R}$. Then:

(i) $\underbrace{f(f(\ldots f(f}_{m}(x_1,\underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-1}),x_2,\underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-2}),\ldots),x_m,\underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-2}) = f(x_1,x_2,\ldots,x_m)$,

(ii) $\underbrace{g(g(\ldots g(g}_{n}(y_1,\underbrace{\mathbf{1},\ldots,\mathbf{1}}_{n-1}),y_2,\underbrace{\mathbf{1},\ldots,\mathbf{1}}_{n-2}),\ldots),y_n,\underbrace{\mathbf{1},\ldots,\mathbf{1}}_{n-2}) = g(y_1,y_2,\ldots,y_n)$.

Proof. (i)

$$\underbrace{f(f(\ldots f(f}_{m}(x_1,\underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-1}),x_2,\underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-2}),\ldots),x_m,\underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-2}). \qquad (2.3)$$

By associativity (Definition 2.1.1 (i)), (2.3) is equal to

$$\underbrace{f(f(\ldots f(f}_{m}(\mathbf{0},\underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-1}),x_1,x_2,\underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-3}),\ldots),x_m,\underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-2})$$

$$=\underbrace{f(f(\ldots f(f}_{m-1}(\mathbf{0},x_1,x_2,\underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-3}),x_3,\underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-2}),\ldots),x_m,\underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-2})$$

$$=\underbrace{f(f(\ldots f(f}_{m-1}(\underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m}),x_1,x_2,x_3,\underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-4}),\ldots),x_m,\underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-2})$$

$$=\underbrace{f(f(\ldots f(f}_{m-2}(\mathbf{0},x_1,x_2,x_3,\underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-4}),\ldots),x_m,\underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-2})$$

$$\vdots$$

$$=f(f(x_1,x_2,\ldots,x_{m-1},\mathbf{0}),x_m,\underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-2})$$

$$=f(f(x_1,x_2,\ldots,x_{m-1},x_m),\underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-1})$$

$$=f(x_1,x_2,\ldots,x_m).$$

(ii) Similar to part (i). □

Lemma 2.7.3. Let $a_1^m, b_2^m, ..., k_2^m \in \mathcal{R}$, t be the number of terms between a_1^m to k_2^m where $t \in \mathbb{Z}_+$ and f be an m-ary operator.

Then by f operation on $a_1, a_2, \ldots, a_m, b_2, b_3, \ldots, b_m, ..., k_2, k_3, \ldots, k_m$ is equal to $\underbrace{f(\ldots f(f}_{t}(a_1^m), b_2^m), \ldots, k_2^m)$.

Proof. By f operation on $a_1, a_2, \ldots, a_m$, we get the following:

$$f(a_1, a_2, \ldots, a_m). \tag{2.4}$$

By f operation on (2.4) with $b_2, b_3, \ldots, b_m$, we get:

$$f(f(a_1, a_2, \ldots, a_m), b_2, b_3, \ldots, b_m) \tag{2.5}$$

By f operation on (2.5) with c_2^m, we get

$$f(f(f(a_1, a_2, \ldots, a_m), b_2, b_3, \ldots, b_m), c_2, c_3, \ldots, c_m)$$

$$\vdots$$

$$\underbrace{f(\ldots(f(f(f}_{t-1}(a_1, a_2, \ldots, a_m), b_2, b_3, \ldots, b_m),$$

$$c_2, c_3, \ldots, c_m)\ldots), j_2, j_3, \ldots, j_m)$$

$$= \underbrace{f(\ldots(f(f(f}_{t-1}(a_1^m), b_2^m), c_2^m)\ldots), j_2^m). \tag{2.6}$$

Similarly, by f operation on (2.6) with k_2^m, we get the following result

$$\underbrace{f(\ldots(f(f(f}_{t}(a_1^m), b_2^m), c_2^m)\ldots), k_2, k_3, \ldots, k_m). \qquad \square$$

Chapter 3

Partial Ordering On Fault Tolerance

In this Chapter we use x_i, y_i, etc., where $i \in \mathbb{Z}_+$ to denote individual system components that are assumed to be *atomic* at the level of discussion, i.e., they have no components or sub-systems of their own. We use *component* to refer to such an atomic part of a system, and *subsystem* to refer to a part of a system that is not necessarily atomic. We assume that components and subsystems are disjoint, in the sense that if fail, they fail independently and do not affect the functioning of other components.

Let $\mathcal{U}$ be a universal set of all systems in the domain of discourse as given by Rao [37], and let f be a mapping $f : \mathcal{U}^m \rightarrow \mathcal{U}$, i.e., f is an m-ary operator. Likewise, let g be an n-ary operator.

3.1 Definition and Example of f and g

Definition 3.1.1. We define f and g for systems as follows:

(i) f is an m-ary operator which applies on systems made up of m components or subsystems, where if any one of the components or subsystems fails, then the whole system fails.

Let a system made up of m components $x_1, x_2, \ldots, x_m$, then the system over operator f is represented as $f(x_1, x_2, \ldots, x_m)$ for all $x_1, x_2, \ldots, x_m \in \mathcal{U}$. The system $f(x_1, x_2, \ldots, x_m)$ fails when any of the components $x_1, x_2, \ldots, x_m$ fails.

(ii) g is an n-ary operator which applies on a system consisting of n components or subsystems, which fails if all the components or subsystems fail; otherwise it continues working even if a single component or subsystem is working properly.

Let a system consist of n components $x_1, x_2, \ldots, x_n$, then the system over operator g is represented as $g(x_1, x_2, \ldots, x_n)$ for all $x_1, x_2, \ldots, x_n \in \mathcal{U}$. The system $g(x_1, x_2, \ldots, x_n)$ fails when all the components $x_1, x_2, \ldots, x_n$ fail.

Example 3.1.2. We mention an example of intrusion detection using wireless sensor networks. The diffusion tree consists of sensor nodes using f and g operations (for details of diffusion tree we can see [3]). The following cases arise.

Case (i) *If a node fails and its predecessor nodes do not join any other nodes in a different sub-diffusion tree dynamically.*

If a node fails then its predecessor nodes are not able to transmit the signal to the next level successor nodes even if they detect an intruder. This is an example of f operation on nodes of the sub-diffusion tree to which the failed node is attached. Due to the failure of one node all its predecessor nodes are unable to take part in communication.

Case (ii) *If a node fails and its predecessor nodes join other nodes in a different sub-diffusion tree dynamically.*

If a node fails then its predecessor nodes join nodes in a different sub-diffusion tree and take part in communication. If they detect an intruder they transmit the signal to their successor nodes and these successor nodes transmit a signal to their successor nodes and so on till it reaches the head node. The head node sends the signal to the base station which does further processing. This case is a g operation on the nodes, where if some nodes fail still these nodes do not affect other nodes in their respective sub-diffusion tree and their predecessor nodes can join another sub-diffusion tree dynamically for transmitting and receiving signals.

Case (iii) *Comparison of fault tolerance behavior of two wireless sensor networks:*

Let two wireless sensor networks consist of m nodes each. If we know the fault tolerance behavior of each node of both networks, and we also know that sensor node n_{1i} of the first network has better fault tolerance than n_{2i} of second network for all $1 \leq i \leq m$, then the first network has better fault tolerance than the second network if they form similar diffusion trees.

3.2 Fault Tolerance Partial Ordering

Consider a partial ordering relation $\preccurlyeq$ on $\mathcal{U}$, such that $(\mathcal{U}, \preccurlyeq)$ is a partially ordered set (poset). This is a *fault-tolerance partial ordering* where $f(x_1^m) \preccurlyeq f(y_1^m)$ means that $f(x_1^m)$ has a lower measure of some fault metric than $f(y_1^m)$ and $f(x_1^m)$ has a better fault tolerance than $f(y_1^m)$, for all $f(x_1^m), f(y_1^m) \in \mathcal{U}$ (see Rao [38] for more details) and $x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m$ are disjoint components.

Assume that **0** represents the atomic system "which is always up" and **1** represents the system "which is always down" (see Rao [38]).

Observation 3.2.1. We observe the following for all disjoint components x_1,$x_2, \ldots, x_m$, $y_1, y_2, \ldots, y_n$, which are in $\mathcal{U}$:

(i) $g(y_1^{j-1}, \mathbf{0}, y_{j+1}^n) = \mathbf{0}$ for all $1 \leq j \leq n$.

This is so since $\mathbf{0}$ represents the component or system which never fails, and as per the definition of g, the system as a whole fails if all the components fail, and otherwise it continues working even if a single component is working properly. In a system $g(y_1^{j-1}, \mathbf{0}, y_{j+1}^n)$, even if all other components y_1^{j-1} and y_{j+1}^n fail even then $\mathbf{0}$ is up and the system is always up.

(ii) $f(x_1^{i-1}, \mathbf{1}, x_{i+1}^m) = \mathbf{1}$ for all $1 \leq i \leq m$.

This is so since $\mathbf{1}$ represents the component or system which is always down, and as per the definition of f if either of the component fails, then the whole system fails. Thus, even though all other components are working properly but due to the component $\mathbf{1}$ the system is always down.

3.3 Partially Ordered (m, n)-Semirings

Definition 3.3.1. If $(\mathcal{U}, f, g)$ is an (m, n)-semiring and $(\mathcal{U}, \preccurlyeq)$ is a poset, then $(\mathcal{U}, f, g, \preccurlyeq)$ is a *partially ordered* (m, n)*-semiring* if the following conditions are satisfied for all $x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n, a, b \in \mathcal{U}$ and $1 \leq i \leq m, 1 \leq j \leq n$.

(i) If $a \preccurlyeq b$, then $f(x_1^{i-1}, a, x_{i+1}^m) \preccurlyeq f(x_1^{i-1}, b, x_{i+1}^m)$.

(ii) If $a \preccurlyeq b$, then $g(y_1^{j-1}, a, y_{j+1}^n) \preccurlyeq g(y_1^{j-1}, b, y_{j+1}^n)$.

Remark 3.3.2. As it is assumed that $\mathbf{0}$ is the system which is always up, it is more fault tolerant than any of the other systems or components. Therefore $\mathbf{0} \preccurlyeq a$, for all $a \in \mathcal{U}$. Similarly, $a \preccurlyeq \mathbf{1}$ because $\mathbf{1}$ is the system that always fails and therefore it is the least fault tolerant; every other system is more fault-tolerant than it.

Observation 3.3.3. The following are obtained for all disjoint components r, s, x_i, y_j, a_i, b_j, which are in $\mathcal{U}$, where $1 \leq i \leq m$, $1 \leq j \leq n$:

(i) $\mathbf{0} \preccurlyeq f(x_1^{i-1}, r, x_{i+1}^m) \preccurlyeq \mathbf{1}$.

(ii) $\mathbf{0} \preccurlyeq g(y_1^{j-1}, s, y_{j+1}^n) \preccurlyeq \mathbf{1}$.

(iii) $\mathbf{0} \preccurlyeq g(y_1^{j-1}, f(a_1^m), y_{j+1}^n) \preccurlyeq \mathbf{1}$.

(iv) $\mathbf{0} \preccurlyeq f(x_1^{i-1}, g(b_1^n), x_{i+1}^m) \preccurlyeq \mathbf{1}$.

From the above description of $\mathbf{0}$ and $\mathbf{1}$, the observation is quite obvious. Case (i) shows that $\mathbf{0}$ is less faulty than $f(x_1^{i-1}, r, x_{i+1}^m)$, and $f(x_1^{i-1}, r, x_{i+1}^m)$ is less faulty than $\mathbf{1}$. Similarly, case (ii) shows that $\mathbf{0}$ is more fault-tolerant than $g(y_1^{j-1}, s, y_{j+1}^n)$ and $g(y_1^{j-1}, s, y_{j+1}^n)$ is more fault-tolerant than $\mathbf{1}$. Likewise, case (iii) shows the operation g over y_1^{j-1}, y_{j+1}^n and f of a_1^m to be less faulty than $\mathbf{1}$ and more faulty than $\mathbf{0}$, and a similar interpretation is made for (iv).

3.4 Main Results

Lemma 3.4.1. If $\preccurlyeq$ is a fault-tolerance partial order and $x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m, z_1, z_2, \ldots, z_n, u_1, u_2, \ldots, u_n$ are disjoint components, which are in $\mathcal{U}$, where $m, n \in \mathbb{Z}_+$, then:

(i) if $x_i \preccurlyeq y_i$, then $f(x_1^m) \preccurlyeq f(y_1^m)$, where $1 \leq i \leq m$ and,

(ii) if $z_j \preccurlyeq u_j$, then $g(z_1^n) \preccurlyeq g(u_1^n)$, where $1 \leq j \leq n$.

Proof. (i) Since $x_i \preccurlyeq y_i$ for all $1 \leq i \leq m$, we have:

$$x_1 \preccurlyeq y_1$$

which is represented as follows:

$$f(\underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-1}, x_1) \preccurlyeq f(\underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-1}, y_1) \tag{3.1}$$

and

$$f(\underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-1}, x_2) \preccurlyeq f(\underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-1}, y_2). \tag{3.2}$$

By f operation on both sides of (3.1) with y_2, we get:

$$f(f(\underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-1}, x_1), y_2, \underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-2}) \preccurlyeq f(f(\underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-1}, y_1), y_2, \underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-2}). \tag{3.3}$$

By f operation on both sides of (3.2) with x_1:

$$f(f(\underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-1}, x_2), x_1, \underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-2}) \preccurlyeq f(f(\underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-1}, y_2), x_1, \underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-2}). \tag{3.4}$$

From (3.3) and (3.4), we get:

$$f(f(\underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-1}, x_1), y_2, \underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-2}) \preccurlyeq f(f(\underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-1}, y_1), y_2, \underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-2}).$$

Similarly, we find for m terms:

$$\begin{aligned}
&\underbrace{f(\dots(f(f}_{m}(\underbrace{\mathbf{0},\dots,\mathbf{0}}_{m-1},x_1),x_2,\underbrace{\mathbf{0},\dots,\mathbf{0}}_{m-2}),\dots),x_m,\underbrace{\mathbf{0},\dots,\mathbf{0}}_{m-2}) \\
&\preccurlyeq \underbrace{f(\dots(f(f}_{m}(\underbrace{\mathbf{0},\dots,\mathbf{0}}_{m-1},y_1),y_2,\underbrace{\mathbf{0},\dots,\mathbf{0}}_{m-2}),\dots),y_m,\underbrace{\mathbf{0},\dots,\mathbf{0}}_{m-2}).
\end{aligned} \tag{3.5}$$

From Lemma 2.7.2, (3.5) may be represented as

$$f(x_1,x_2,\dots,x_m) \preccurlyeq f(y_1,y_2,\dots,y_m)$$

so

$$f(x_1^m) \preccurlyeq f(y_1^m).$$

(ii) Since $z_j \preccurlyeq y_j$, for all $1 \le j \le n$

$$g(\underbrace{\mathbf{1},\dots,\mathbf{1}}_{n-1},z_1) \preccurlyeq g(\underbrace{\mathbf{1},\dots,\mathbf{1}}_{n-1},u_1)$$

and

$$g(\underbrace{\mathbf{1},\dots,\mathbf{1}}_{n-1},z_2) \preccurlyeq g(\underbrace{\mathbf{1},\dots,\mathbf{1}}_{n-1},u_2).$$

After following similar steps as seen in part (i), we use the g operation for n terms,

$$g(\underbrace{\ldots(g(g}_{n}(\underbrace{\mathbf{1},\ldots,\mathbf{1}}_{n-1},z_1),z_2,\underbrace{\mathbf{1},\ldots,\mathbf{1}}_{n-2}),\ldots),z_n,\underbrace{\mathbf{1},\ldots,\mathbf{1}}_{n-2})$$

$$\preccurlyeq \underbrace{g(\ldots(g(g}_{n}(\underbrace{\mathbf{1},\ldots,\mathbf{1}}_{n-1},u_1),u_2,\underbrace{\mathbf{1},\ldots,\mathbf{1}}_{n-2}),\ldots),u_n,\underbrace{\mathbf{1},\ldots,\mathbf{1}}_{n-2})$$

which is represented as

$$g(z_1,z_2,\ldots,z_n) \preccurlyeq g(u_1,u_2,\ldots,u_n)$$

and so

$$g(z_1^n) \preccurlyeq g(u_1^n).$$

□

Theorem 3.4.2. If $\preccurlyeq$ is a fault-tolerance partial order and given disjoint components a_i, c_j, b_i, d_j in $\mathcal{U}$, where $1 \le i \le m$, $1 \le j \le n$, the following obtain:

(i) If $a_i \preccurlyeq b_i$, where $1 \le i \le m$, then:

$g(y_1^{j-1}, f(a_1^m), y_{j+1}^n) \preccurlyeq g(y_1^{j-1}, f(b_1^m), y_{j+1}^n)$, for all $y_1, y_2, \ldots, y_n \in \mathcal{U}$ and $1 \le j \le n$.

(ii) If $c_j \preccurlyeq d_j$, where $1 \le j \le n$, then:

$f(x_1^{k-1}, g(c_1^n), x_{k+1}^m) \preccurlyeq f(x_1^{k-1}, g(d_1^n), x_{k+1}^m)$, for all $x_1, x_2, \ldots, x_m \in \mathcal{U}$ and $1 \le k \le m$.

Proof. (i) Since $a_i \preccurlyeq b_i$, for all $1 \le i \le m$.

Therefore, from Lemma 3.4.1 (i)

$$f(a_1^m) \preccurlyeq f(b_1^m), \qquad \forall a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_m \in \mathcal{U}.$$

From Definition 3.3.1 of a partially ordered (m, n)-semiring, we deduce that

$$g(y_1^{j-1}, f(a_1^m), y_{j+1}^n) \preccurlyeq g(y_1^{j-1}, f(b_1^m), y_{j+1}^n)$$

for all $1 \leq j \leq n$.

(ii) Since $c_j \preccurlyeq d_j$, for all $1 \leq j \leq n$, from Lemma 3.4.1 (ii), we find that

$$g(c_1^n) \preccurlyeq g(d_1^n), \qquad \forall c_1, c_2, \ldots, c_n, d_1, d_2, \ldots, d_n \in \mathcal{U}.$$

From Definition 3.3.1 of a partially ordered (m, n)-semiring, we deduce that

$$f(x_1^{k-1}, g(c_1^n), x_{k+1}^m) \preccurlyeq f(x_1^{k-1}, g(d_1^n), x_{k+1}^m)$$

for all $1 \leq k \leq m$. □

Lemma 3.4.3. If $\preccurlyeq$ is a fault-tolerance partial order and x_i, y_j are disjoint components which are in $\mathcal{U}$, where $1 \leq i \leq m$ and $1 \leq j \leq n$, we get the following:

(i) $x_i \preccurlyeq f(x_1, x_2, \ldots, x_m)$,

(ii) $g(y_1, y_2, \ldots, y_n) \preccurlyeq y_j$.

Proof. (i) As

$$\mathbf{0} \preccurlyeq x_1, \tag{3.6}$$

by f operation on both sides of (3.6) with x_i, we get

$$f(\mathbf{0}, x_i, \underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-2}) \preccurlyeq f(x_1, x_i, \underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-2}).$$

Therefore,

$$x_i \preccurlyeq f(x_1, x_i, \underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-2}).$$

Similarly, we obtain:

$$\begin{aligned} x_i \preccurlyeq f(x_1, x_i, \underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-2}) \preccurlyeq \ldots \preccurlyeq f(x_1, x_2, x_i, \ldots, x_{m-1}, \mathbf{0}) \\ \preccurlyeq f(x_1, x_2, \ldots, x_m). \end{aligned} \tag{3.7}$$

Hence,

$$x_i \preccurlyeq f(x_1, x_2, \ldots, x_m)$$

for all $1 \leq i \leq m$.

(ii) As

$$y_1 \preccurlyeq \mathbf{1}, \tag{3.8}$$

by g operation on both sides of (3.8) with y_j, we get

$$g(y_1, y_j, \underbrace{\mathbf{1}, \ldots, \mathbf{1}}_{n-2}) \preccurlyeq y_j.$$

Similarly, we obtain:

$$g(y_1, y_2, \ldots, y_n) \preccurlyeq g(y_1, y_2, y_j, \ldots, y_{n-1}, \mathbf{1}) \preccurlyeq \ldots \preccurlyeq \\ g(y_1, y_j, \underbrace{\mathbf{1}, \ldots, \mathbf{1}}_{n-2}) \preccurlyeq y_j. \tag{3.9}$$

Hence,

$$g(y_1, y_2, \ldots, y_n) \preccurlyeq y_j$$

for all $1 \leq j \leq n$. □

Corollary 3.4.4. If $\preccurlyeq$ is a fault-tolerance partial order, then the following hold for all disjoint components x_i, y_j which are elements of $\mathcal{U}$, where $1 \leq i \leq m$, $1 \leq j \leq n$ and $k, t \in \mathbb{Z}_+$:

(i) $f(x_1, x_2, \ldots, x_k, \underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-k}) \preccurlyeq f(x_1^m)$,

where $k < m$; and

(ii) $g(y_1^n) \preccurlyeq g(y_1, y_2, \ldots, y_t, \underbrace{\mathbf{1}, \ldots, \mathbf{1}}_{n-t})$,

where $t < n$.

Proof. (i) From (3.7) we deduce that,

$$f(x_1, \ldots, x_k, \underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-k}) \preccurlyeq f(x_1, \ldots, x_{k+1}, \underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-k-1}) \\ \preccurlyeq \ldots \preccurlyeq f(x_1, x_2, \ldots, x_m).$$

Therefore,

$$f(x_1, \ldots, x_k, \underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-k}) \preccurlyeq f(x_1^m).$$

(ii) As in part (i), we deduce from (3.9) that:

$$g(y_1^n) \preccurlyeq g(y_1, y_2, \ldots, y_t, \underbrace{\mathbf{1}, \ldots, \mathbf{1}}_{n-t}).$$

□

Theorem 3.4.5. If $\preccurlyeq$ is a fault-tolerance partial order and $x_1, x_2, \ldots, x_m$,$y_1, y_2, \ldots, y_m$,$z_1, z_2, \ldots, z_n$, $u_1, u_2, \ldots, u_n$ are disjoint components in $\mathcal{U}$, then following hold:

(i) If $f(x_i, \ldots, x_m, \underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{i-1}) \preccurlyeq f(y_i, \ldots, y_m, \underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{i-1})$,

then $f(x_1, x_2, \ldots, x_m) \preccurlyeq f(y_1, y_2, \ldots, y_m)$ for all $1 < i < m$.

(ii) If $g(z_j, \ldots, z_n, \underbrace{\mathbf{1}, \ldots, \mathbf{1}}_{j-1}) \preccurlyeq g(u_j, \ldots, u_n, \underbrace{\mathbf{1}, \ldots, \mathbf{1}}_{j-1})$,

then $g(z_1, z_2, \ldots, z_n) \preccurlyeq g(u_1, u_2, \ldots, u_n)$ for all $1 < j < n$.

$f(\overset{(m)}{f(a_1^m)})$ represents the system which is obtained after applying the f operation on m repeated $f(a_1^m)$ systems or subsystems. Similarly, $g(\overset{(n)}{g(b_1^n)})$ represents the system which is obtained after applying the g operation on n repeated $g(b_1^n)$ systems or subsystems.

Theorem 3.4.6. If $\preccurlyeq$ is a fault-tolerance partial order, and components $x_1, x_2, \ldots, x_m$,$y_1, y_2, \ldots, y_n$ are disjoint components and are in $\mathcal{U}$, then:

(i) $f(x_1^m) \preccurlyeq f(\overset{(m)}{f(x_1^m)})$,

(ii) $g(\overset{(n)}{g(y_1^n)}) \preccurlyeq g(y_1^n)$.

Proof. (i) **0** represents the system which is always up, which is more fault tolerant than any other system. Hence it is more fault tolerant than $f(x_1^m)$, i.e.,

$$\mathbf{0} \preccurlyeq f(x_1^m), \tag{3.10}$$

so by f operation on both sides of (3.10) with $f(x_1^m)$, we get

$$f(\mathbf{0}, f(x_1^m), \underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-2}) \preccurlyeq f(f(x_1^m), f(x_1^m), \underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-2})$$

which is written as

$$f(x_1^m) \preccurlyeq f(\underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-2}, \overset{(2)}{f}(x_1^m)).$$

Similarly, we get the following:

$$\mathbf{0} \preccurlyeq f(\underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-1}, f(x_1^m)) \preccurlyeq \ldots \preccurlyeq f(\mathbf{0}, \overset{(m-1)}{f}(x_1^m)) \preccurlyeq f(\overset{(m)}{f}(x_1^m)).$$

Thus, we deduce that

$$f(x_1^m) \preccurlyeq f(\overset{(m)}{f}(x_1^m)).$$

(ii) **1** represents the system which is always down, therefore any other system is more fault tolerant than **1**. Hence $g(y_1^n)$ is more fault tolerant than **1**. Therefore,

$$g(y_1^n) \preccurlyeq \mathbf{1}, \tag{3.11}$$

and by g operation on both sides of (3.11) with $g(y_1^n)$, we get

$$g(\underbrace{\mathbf{1},\ldots,\mathbf{1}}_{n-2},\overset{(2)}{g}(y_1^n)) \preccurlyeq g(y_1^n).$$

Similarly, we deduce the following:

$$g(\overset{(n)}{g}(y_1^n)) \preccurlyeq g(\mathbf{1},\overset{(n-1)}{g}(y_1^n)). \tag{3.12}$$

From (3.12), we get

$$g(\overset{(n)}{g}(y_1^n)) \preccurlyeq g(\mathbf{1},\overset{(n-1)}{g}(y_1^n)) \preccurlyeq,\ldots,\preccurlyeq g(y_1^n).$$

Thus, we deduce the following

$$g(\overset{(n)}{g}(y_1^n)) \preccurlyeq g(y_1^n). \qquad \square$$

Corollary 3.4.7. The following hold for all disjoint components $x_1,\ldots,x_m$,$z_1,\ldots,z_n$, $y_1,\ldots,y_m$, $u_1,\ldots,u_n$, which are elements of $\mathcal{U}$, where $m,n \in \mathbb{Z}_+$:

(i) If $f(\overset{(m)}{f}(x_1^m)) \preccurlyeq f(y_1^m)$, then

$f(x_1^m) \preccurlyeq f(y_1^m)$.

(ii) If $g(z_1^n) \preccurlyeq g(\overset{(n)}{g}(u_1^n))$, then

$g(z_1^n) \preccurlyeq g(u_1^n)$.

Proof. (i) $f(\overset{(m)}{f}(x_1^m)) \preccurlyeq f(y_1^m)$

and from Theorem 3.4.6, $f(x_1^m) \preccurlyeq f(\overset{(m)}{f}(x_1^m))$.

Therefore, $f(x_1^m) \preccurlyeq f(y_1^m)$.

(ii) The proof is very similar to that of part (i). □

Corollary 3.4.8. Let k and t be positive integers and $k < m$, $t < n$. Given disjoint components $x_1, \dots, x_m$, $y_1, y_2, \dots, y_m$, $z_1, z_2, \dots, z_n$,$u_1, u_2, \dots, u_n$ that are in $\mathcal{U}$, the following hold:

(i) If $f(\underbrace{\mathbf{0}, \dots, \mathbf{0}}_{m-k}, \overset{(k)}{f}(x_1^m)) \preccurlyeq f(y_1^m)$, then $f(x_1^m) \preccurlyeq f(y_1^m)$.

(ii) If $g(z_1^n) \preccurlyeq g(\underbrace{\mathbf{1}, \dots, \mathbf{1}}_{n-t}, \overset{(t)}{g}(u_1^n))$, then $g(z_1^n) \preccurlyeq g(u_1^n)$.

Proof. Similar to Corollary 3.4.7. □

Theorem 3.4.9. Let $\preccurlyeq$ be a fault-tolerance partial order and $x_i \preccurlyeq y_i$ and $z_j \preccurlyeq u_j$ for all $x_i, y_i, z_j, u_j \in \mathcal{U}$, where $1 \leq i \leq m$ and $1 \leq j \leq n$. Then the following obtain:

(i) $f(\overset{(m)}{f}(x_1^m)) \preccurlyeq f(\overset{(m)}{f}(y_1^m))$,

(ii) $g(\overset{(n)}{g}(z_1^n)) \preccurlyeq g(\overset{(n)}{g}(u_1^n))$,

(iii) $f(\overset{(m)}{g}(z_1^n)) \preccurlyeq f(\overset{(m)}{g}(u_1^n))$,

(iv) $g(\overset{(n)}{f}(x_1^m)) \preccurlyeq g(\overset{(n)}{f}(y_1^m))$.

Proof. (i) As

$$x_i \preccurlyeq y_i, \qquad 1 \leq i \leq m,$$

from Lemma 3.4.1 (i), we get

$$f(x_1^m) \preccurlyeq f(y_1^m).$$

This is written as

$$f(\underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-1}, f(x_1^m)) \preccurlyeq f(\underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-1}, f(y_1^m)). \tag{3.13}$$

So by f operation on both sides of (3.13) with $f(x_1^m)$, we get,

$$\begin{aligned} f(f(\underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-1}, f(x_1^m)), f(x_1^m), \underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-2}) \preccurlyeq \\ f(f(\underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-1}, f(x_1^m)), f(y_1^m), \underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-2}). \end{aligned} \tag{3.14}$$

So by f operation on both sides of (3.13) with $f(y_1^m)$, we get,

$$\begin{aligned} f(f(\underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-1}, f(x_1^m)), f(y_1^m), \underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-2}) \preccurlyeq \\ f(f(\underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-1}, f(y_1^m)), \underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-2}, f(y_1^m)). \end{aligned} \tag{3.15}$$

From (3.14) and (3.15), we get,

$$f(\underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-2}, \overset{(2)}{f}(x_1^m)) \preccurlyeq f(\underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-2}, \overset{(2)}{f}(y_1^m)).$$

Similarly, we get for m terms:

$$f(\overset{(m)}{f}(x_1^m)) \preccurlyeq f(\overset{(m)}{f}(y_1^m)).$$

(ii) We know that

$$z_j \preccurlyeq u_j, \qquad 1 \le j \le n.$$

From Lemma 3.4.1 (ii), we get,

$$g(z_1^n) \preccurlyeq g(u_1^n).$$

Which is represented as following

$$g(\underbrace{\mathbf{1},\ldots,\mathbf{1}}_{n-1}, g(z_1^n)) \preccurlyeq g(\underbrace{\mathbf{1},\ldots,\mathbf{1}}_{n-1}, g(u_1^n)). \tag{3.16}$$

Now by g operation on both sides of (3.16) with $g(z_1^n)$, we get,

$$\overset{(2)}{g}(g(z_1^n), \underbrace{\mathbf{1},\ldots,\mathbf{1}}_{n-2}) \preccurlyeq g(g(z_1^n), g(u_1^n), \underbrace{\mathbf{1},\ldots,\mathbf{1}}_{n-2}). \tag{3.17}$$

So by g operation on both sides of (3.16) with $g(u_1^n)$, we get,

$$g(\underbrace{\mathbf{1},\ldots,\mathbf{1}}_{n-2}, g(z_1^n), g(u_1^n)) \preccurlyeq g(\underbrace{\mathbf{1},\ldots,\mathbf{1}}_{n-2}, \overset{(2)}{g}(u_1^n)). \tag{3.18}$$

So now from (3.17) and (3.18), we get,

$$g(\underbrace{\mathbf{1},\ldots,\mathbf{1}}_{n-2}, \overset{(2)}{g}(z_1^n)) \preccurlyeq g(\underbrace{\mathbf{1},\ldots,\mathbf{1}}_{n-2}, \overset{(2)}{f}(u_1^n)).$$

Similarly, we find for n terms

$$g(\overset{(n)}{g}(z_1^n)) \preccurlyeq g(\overset{(n)}{g}(u_1^n)).$$

(iii) From Lemma 3.4.1 (ii)

$$g(z_1^n) \preccurlyeq g(u_1^n).$$

Similar to part (i), we find f of m terms and get

$$f(\underbrace{g(z_1^n), g(z_1^n), \ldots, g(z_1^n)}_{m}) \preccurlyeq f(\underbrace{g(u_1^n), g(u_1^n), \ldots, g(u_1^n)}_{m}),$$

$$\overset{(m)}{f(g}(z_1^n)) \preccurlyeq \overset{(m)}{f(g}(u_1^n)).$$

(iv) We know that

$$x_i \preccurlyeq y_i, \qquad 1 \leq i \leq m,$$

so from Lemma 3.4.1 (i), we get

$$f(x_1^m) \preccurlyeq f(y_1^m).$$

As proved in part (ii), we find g of n terms and get,

$$g(\underbrace{f(x_1^m), f(x_1^m), \ldots, f(x_1^m)}_{n}) \preccurlyeq g(\underbrace{f(y_1^m), f(y_1^m), \ldots, f(y_1^m)}_{n}).$$

Thus, we get,

$$\overset{(n)}{g(f(}x_1^m)) \preccurlyeq \overset{(n)}{g(f(}y_1^m)) \ . \qquad \square$$

Corollary 3.4.10. If $\preccurlyeq$ is a fault-tolerance partial order and $k < m$, $t < n$ where $k, t \in \mathbb{Z}_+$, if $x_i \preccurlyeq y_i$, $z_j \preccurlyeq u_j$ for all disjoint components x_i, z_j, y_i, u_j, which are in $\mathcal{U}$, where $1 \leq i \leq m$ and $1 \leq j \leq n$ then:

(i) $f(\underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-k}, f(\overset{(k)}{x_1^m})) \preccurlyeq f(\underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-k}, f(\overset{(k)}{y_1^m}))$

(ii) $g(\underbrace{\mathbf{1},\ldots,\mathbf{1}}_{n-t}\, g(\overset{(t)}{z_1^n})) \preccurlyeq g(\underbrace{\mathbf{1},\ldots,\mathbf{1}}_{n-t}, g(\overset{(t)}{u_1^n}))$.

Proof. (i) As $x_i \preccurlyeq y_i, \quad 1 \leq i \leq m$ and f is an m-ary operator, we get from Lemma 3.4.1 (i),

$$f(x_1^m) \preccurlyeq f(y_1^m).$$

As proved in Theorem 3.4.9 (i), we find f of k terms $\forall k \in \mathbb{Z}_+$, and $k < m$, we obtain

$$f(\underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-k}, f(\overset{(k)}{x_1^m})) \preccurlyeq f(\underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-k}, f(\overset{(k)}{y_1^m})).$$

(ii) As $z_j \preccurlyeq u_j$, $1 \leq j \leq n$ and g is an n-ary operator, from Lemma 3.4.1 (ii), we get,

$$g(z_1^n) \preccurlyeq g(u_1^n).$$

As proved in Theorem 3.4.9 (ii), we find the g of t terms $\forall t \in \mathbb{Z}_+$, where $t < n$, so that

$$g(\underbrace{\mathbf{1},\ldots,\mathbf{1}}_{n-t}, g(\overset{(t)}{z_1^n})) \preccurlyeq g(\underbrace{\mathbf{1},\ldots,\mathbf{1}}_{n-t}, g(\overset{(t)}{u_1^n})).$$ □

Theorem 3.4.11. If $\preccurlyeq$ is a fault-tolerance partial order, disjoint components a_i, b_i, c_j, d_j, x_k, y_k, z_t, u_t are in $\mathcal{U}$ and $a_i \preccurlyeq b_i, c_j \preccurlyeq d_j$, $x_k \preccurlyeq y_k$ and $z_t \preccurlyeq u_t$, where $1 \leq i \leq m, 1 \leq j \leq n, 1 \leq k \leq m$ and $1 \leq t \leq n$, then:

(i) $f(x_1^{k-1}, f(a_1^m), x_{k+1}^m) \preccurlyeq f(y_1^{k-1}, f(b_1^m), y_{k+1}^m)$,

for all $1 \leq k \leq m$; and

(ii) $f(x_1^{k-1}, g(c_1^n), x_{k+1}^m) \preccurlyeq f(y_1^{k-1}, g(d_1^n), y_{k+1}^m)$,

for all $1 \leq k \leq m$; and

(iii) $g(z_1^{t-1}, f(a_1^m), z_{t+1}^n) \preccurlyeq g(u_1^{t-1}, f(b_1^m), u_{t+1}^n)$,

for all $1 \leq t \leq n$; and

(iv) $g(z_1^{t-1}, g(c_1^n), z_{t+1}^n) \preccurlyeq g(u_1^{t-1}, g(d_1^n), u_{t+1}^n)$,

for all $1 \leq t \leq n$.

Proof. (i) From Lemma 3.4.1 (i), if $a_i \preccurlyeq b_i$, then $f(a_1^m) \preccurlyeq f(b_1^m)$, $1 \leq i \leq m$.

We prove in a similar manner as Lemma 3.4.1 (i) that

$$f(f(a_1^m), x_1, \underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-2}) \preccurlyeq f(f(b_1^m), y_1, \underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-2}).$$

Similarly, we get,

$$f(f(a_1^m), x_1^{k-1}, x_{k+1}^m) \preccurlyeq f(f(b_1^m), y_1^{k-1}, y_{k+1}^m).$$

Thus,

$$f(x_1^{k-1}, f(a_1^m), x_{k+1}^m) \preccurlyeq f(y_1^{k-1}, f(b_1^m), y_{k+1}^m).$$

Similar to the above, we can prove (ii), (iii) and (iv). $\square$

Chapter 4

Fault Tolerance In Congruent Systems

4.1 Background

A relation is called an equivalence relation if it is reflexive, symmetric and transitive [28].

Definition 4.1.1. (i) An equivalence relation $\cong$ on the elements of a semigroup $(\mathcal{U}, +)$ is called a *congruence relation* if it satisfies following:

if $x_1 \cong y_1$ and $x_2 \cong y_2$ then $x_1 + x_2 \cong y_1 + y_2$.

(ii) An equivalence relation $\cong$ on the elements of a semiring $(\mathcal{U}, +, \times)$ is called a *semiring congruence relation* if it satisfies following:

if $x_1 \cong y_1$ and $x_2 \cong y_2$ then $x_1 + x_2 \cong y_1 + y_2$

and $x_1 \times x_2 \cong y_1 \times y_2$

for all $x_1, x_2, y_1, y_2 \in \mathcal{U}$ [27].

Given by Hadjicostis [27], and Hebisch and Weinert [28].

We generalize this relation for f and g which are m-ary and n-ary operators respectively, as $(\mathcal{U}, f, g)$ (m, n)-semiring congruence relation.

4.2 (m, n)-Semiring Congruence Relation

Definition 4.2.1. Let $(\mathcal{U}, f, g)$ be an (m, n)-semiring and $\cong$ be an equivalence relation on $\mathcal{U}$. Then $\cong$ is an *(m, n)-semiring congruence relation*,

(i) If $x_i \cong y_i$ when $f(x_1^m) \cong f(y_1^m)$ for all $1 \leq i \leq m$ and

(ii) If $z_j \cong u_j$ when $g(z_1^n) \cong g(u_1^n)$ for all $1 \leq j \leq n$

for all $x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m$,$z_1, z_2, \ldots, z_n$,$u_1, u_2, \ldots, u_n \in \mathcal{U}$.

We have generalized the definition given by Hadjicostis [27] and Hebisch and Weinert [28] to get Definition 4.2.1, for more details we can see [10].

We use congruence relation in place of (m, n)-semiring congruence relation for simplicity.

If two systems are congruent then their fault tolerance behavior will be similar.

Following Lemma 4.2.2 states that if components a and b are related by congruence related $\cong$, which shows that the fault tolerance metric of a and b are similar. Then if we apply f operation on a and b with components x_1^{i-1} and x_{i+1}^m, then we get the systems which have similar fault tolerance behavior. Similar is the case if we apply g operation.

Lemma 4.2.2. Let $\cong$ be a congruence relation and $a, b, x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n$ be disjoint components and are in $\mathcal{U}$. If $a \cong b$ and $a, b \in \mathcal{U}$ then

(i) $f(x_1^{i-1}, a, x_{i+1}^m) \cong f(x_1^{i-1}, b, x_{i+1}^m)$,

where $1 \leq i \leq m$.

(ii) $g(y_1^{j-1}, a, y_{j+1}^n) \cong g(y_1^{j-1}, b, y_{j+1}^n)$,

where $1 \leq j \leq n$.

Proof. (i) As

$$a \cong b \qquad a, b \in \mathcal{U} \tag{4.1}$$

by f operation on both sides of (4.1) with $f(x_1^{i-1}, \mathbf{0}, x_{i+1}^m)$, we get

$$f(f(x_1^{i-1}, \mathbf{0}, x_{i+1}^m), a, \underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-2}) \cong f(f(x_1^{i-1}, \mathbf{0}, x_{i+1}^m), b, \underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-2})$$

which can be written as following:

$$f(f(x_1^{i-1}, a, x_{i+1}^m), \underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-1}) \cong f(f(x_1^{i-1}, b, x_{i+1}^m), \underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-1})$$

by using the property of f-identity element $\mathbf{0}$, we get the following result:

$$f(x_1^{i-1}, a, x_{i+1}^m) \cong f(x_1^{i-1}, b, x_{i+1}^m)$$

for all $a, b \in \mathcal{U}$ and $1 \leq i \leq m$.

(ii)

$$a \cong b \qquad \forall a, b \in \mathcal{U} \tag{4.2}$$

by g operation on both sides of (4.2) with $g(y_1^{j-1}, \mathbf{1}, y_{j+1}^n)$, we get

$$g(g(y_1^{j-1}, \mathbf{1}, y_{j+1}^n), a, \underbrace{\mathbf{1}, \ldots, \mathbf{1}}_{n-2}) \cong g(g(y_1^{j-1}, \mathbf{1}, y_{j+1}^n), b, \underbrace{\mathbf{1}, \ldots, \mathbf{1}}_{n-2})$$

is written as:

$$g(g(y_1^{j-1}, a, y_{j+1}^n), \underbrace{\mathbf{1}, \ldots, \mathbf{1}}_{n-1}) \cong g(g(y_1^{j-1}, b, y_{j+1}^n), \underbrace{\mathbf{1}, \ldots, \mathbf{1}}_{n-1})$$

by using the property of g-identity element $\mathbf{1}$, we get the following result:

$$g(y_1^{j-1}, a, y_{j+1}^n) \cong g(y_1^{j-1}, b, y_{j+1}^n)$$

, $1 \leq j \leq n$. □

Theorem 4.2.3. If $\cong$ is a congruence relation and $x_1, x_2, \ldots, x_m,\ y_1, y_2, \ldots, y_m,$ $z_1, z_2, \ldots, z_n,\ u_1, u_2, \ldots, u_n$ be disjoint components and be in $\mathcal{U}$ then following hold

(i) If $f(\overset{(m)}{f}(x_1^m)) \cong f(y_1^m)$ then $f(x_1^m) \cong y_i$

for all $1 \leq i \leq m$.

(ii) If $g(\overset{(n)}{g}(z_1^n)) \cong g(u_1^n)$ then $g(z_1^n) \cong u_j$

for all $1 \leq j \leq n$.

Proof. (i) As

$$f(\overset{(m)}{f}(x_1^m)) \cong f(y_1^m)$$

which is written as

$$f(\underbrace{f(x_1^m), \ldots, f(x_1^m)}_{m}) \cong f(y_1^m).$$

Thus,

$$y_1 = y_2 = \ldots = y_m \cong f(x_1^m)$$

Therefore, $y_i \cong f(x_1^m)$.

(ii) Similar to (i).

□

Theorem 4.2.4. If $x_i \cong y_i$, $z_j \cong u_j$ and $x_1, x_2, \ldots, x_m$, $y_1, y_2, \ldots, y_m$, $z_1, z_2, \ldots, z_n$, $u_1, u_2, \ldots, u_n$ then

(i) for every $1 \le i \le m$

$$f(\overset{(m)}{f}(x_1^m)) \cong f(\overset{(m)}{f}(y_1^m))$$

(ii) for every $1 \le j \le n$

$$g(\overset{(n)}{g}(z_1^n)) \cong g(\overset{(n)}{g}(u_1^n))$$

Proof. (i) As we know

$f(x_1^m) \cong f(y_1^m)$ if $x_i \cong y_i$ for all $1 \le i \le m$.

As $f(a_1^m) \cong f(b_1^m)$ is represented as

$$f(a_1, a_2, \ldots, a_m) \cong f(b_1, b_2, \ldots, b_m) \tag{4.3}$$

if $a_1 = a_2 = \ldots = a_m = f(x_1^m)$ and $b_1 = b_2 = \ldots = b_m = f(y_1^m)$

Then, (4.3) can be written as

$$f(\underbrace{f(x_1^m), f(x_1^m), \ldots, f(x_1^m)}_{m}) \cong f(\underbrace{f(y_1^m), f(y_1^m), \ldots, f(y_1^m)}_{m})$$

which is represented as following

$$f(\overset{(m)}{f}(x_1^m)) \cong f(\overset{(m)}{f}(y_1^m)$$

for all $x_1^m, y_1^m \in \mathcal{U}$ and $1 \leq i \leq m$.

(ii) In similar manner we can prove

$$g(\overset{(n)}{g}(z_1^n)) \cong g(\overset{(n)}{g}(u_1^n))$$

, where $1 \leq j \leq n$. □

Theorem 4.2.5. If $x_i \cong y_i$, $z_j \cong u_j$ and $x_1, x_2, \ldots, x_m$, $y_1, y_2, \ldots, y_m$, $z_1, z_2, \ldots, z_n$, $u_1, u_2, \ldots, u_n$ then we have

(i) for every $1 \leq j \leq n$

$$f(\overset{(m)}{g}(z_1^n)) \cong f(\overset{(m)}{g}(u_1^n))$$

(ii) for every $1 \leq i \leq m$

$$g(\overset{(n)}{f}(x_1^m)) \cong g(\overset{(n)}{f}(y_1^m))$$

Proof. (i) As we know

$g(z_1^n) \cong g(u_1^n)$ if $z_j \cong u_j$ for all $1 \leq j \leq n$.

Let $f(a_1^m) \cong f(b_1^m)$ which is represented as

$$f(a_1, a_2, \ldots, a_m) \cong f(b_1, b_2, \ldots, b_m)$$

if $a_1 = a_2 = \ldots = a_m = g(z_1^n)$ and $b_1 = b_2 = \ldots = b_m = g(u_1^n)$. Then we get the following result

$$f(\underbrace{g(z_1^n), g(z_1^n), \ldots, g(z_1^n)}_{m}) \cong f(\underbrace{g(u_1^n), g(u_1^n), \ldots, g(u_1^n)}_{m})$$

which is represented as follows

$$f(\overset{(m)}{g(z_1^n)}) \cong f(\overset{(m)}{g(u_1^n)})$$

for all $1 \leq i \leq n$.

(ii) In a similar manner we can prove

$$g(\overset{(n)}{f(x_1^m)}) \cong g(\overset{(n)}{f(y_1^m)})$$

,where $1 \leq j \leq m$. □

Lemma 4.2.6. If $a \cong b$ and $x_i \cong y_i$, $z_j \cong u_j$ and $a, b \in \mathcal{U}$ then

(i)

$$f(x_1^{i-1}, a, x_{i+1}^m) \cong f(y_1^{i-1}, b, y_{i+1}^m)$$

for all $x_1, x_2, \ldots, x_m$, $y_1, y_2, \ldots, y_m \in \mathcal{U}$ and $1 \leq i \leq m$.

(ii)

$$g(z_1^{j-1}, a, z_{j+1}^n) \cong g(u_1^{j-1}, b, u_{j+1}^n)$$

for all $z_1, z_2, \ldots, z_n, u_1, u_2, \ldots, u_n \in \mathcal{U}$ and $1 \leq j \leq n$.

Proof. (i) From above Definition 4.2.1 of (m, n)-semiring congruence relation $f(x_1^m) \cong f(y_1^m)$ if $x_i \cong y_i$ for all $1 \leq i \leq m$. Which is written as following

$$f(x_1^{i-1}, x_i, x_{i+1}^m) \cong f(y_1^{i-1}, y_i, y_{i+1}^m) \tag{4.4}$$

replace x_i by a and y_i by b in (4.4) as $a \cong b$, to get

$$f(x_1^{i-1}, a, x_{i+1}^m) \cong f(y_1^{i-1}, b, y_{i+1}^m)$$

for all $a, b \in \mathcal{U}$ and $1 \leq i \leq m$.

(ii) From Definition 4.2.1 of (m, n)-semiring congruence relation

$g(z_1^n) \cong g(u_1^n)$ if $z_j \cong u_j$ for all $1 \leq j \leq n$.

We write the equation as following

$$g(z_1^{j-1}, z_j, z_{j+1}^n) \cong g(u_1^{j-1}, u_j, u_{j+1}^n) \tag{4.5}$$

As $a \cong b$, replace z_j by a and u_j by b in (4.5) to get the following result

$$g(z_1^{j-1}, a, z_{j+1}^n) \cong g(u_1^{j-1}, b, u_{j+1}^n)$$

for all $a, b \in \mathcal{U}$ and $1 \leq j \leq n$. □

Lemma 4.2.7. If $\cong$ is a congruence relation and $x_i \cong y_i$, $z_j \cong u_j$ $\forall x_i, y_i, z_j, u_j \in \mathcal{U}$ then

(i) $f(f(\overset{(i)}{x_1^m}), \underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-i}) \cong f(f(\overset{(i)}{y_1^m}), \underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-i})$

for all $1 \leq i \leq m$.

(ii) $g(g(\overset{(j)}{z_1^n}), \underbrace{\mathbf{1}, \ldots, \mathbf{1}}_{n-j}) \cong g(g(\overset{(j)}{u_1^n}), \underbrace{\mathbf{1}, \ldots, \mathbf{1}}_{n-j})$

for all $1 \leq j \leq n$.

(iii) $f(g(\overset{(i)}{z_1^n}), \underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-i}) \cong f(g(\overset{(i)}{u_1^n}), \underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-i})$.

for all $1 \leq i \leq m$.

(iv) $g(f(\overset{(j)}{x_1^m}), \underbrace{\mathbf{1}, \ldots, \mathbf{1}}_{n-j}) \cong g(f(\overset{(j)}{y_1^m}), \underbrace{\mathbf{1}, \ldots, \mathbf{1}}_{n-j})$

for all $1 \leq j \leq n$.

Proof. (i) As

$$x_i \cong y_i$$

for all $1 \leq i \leq n$.

Then, from Definition 4.2.1

$$f(x_1^m) \cong f(y_1^m) \tag{4.6}$$

by f operation on both sides of (4.6) with $f(x_1^m)$, we get

$$f(f(x_1^m), f(x_1^m), \underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-2}) \cong f(f(y_1^m), f(x_1^m), \underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-2}).$$

which is written as follows

$$f(f(\overset{(2)}{x_1^m}), \underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-2}) \cong f(f(y_1^m), f(x_1^m), \underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-2}) \tag{4.7}$$

by f operation on both sides of (4.6) with $f(y_1^m)$, we get the following result

$$f(f(x_1^m), f(y_1^m), \underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-2}) \cong f(f(y_1^m), f(y_1^m), \underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-2}).$$

Which can be written as following

$$f(f(x_1^m), f(y_1^m), \underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-2}) \cong f(f(\overset{(2)}{y_1^m}), \underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-2}) \tag{4.8}$$

From (4.7) and (4.8), we get

$$f(f(\overset{(2)}{x_1^m}), \underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-2}) \cong f(f(\overset{(2)}{y_1^m}), \underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-2}).$$

Similarly, we can find for i terms

$$f(f(\overset{(i)}{x_1^m}), \underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-i}) \cong f(f(\overset{(i)}{y_1^m}), \underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-i})$$

(ii) Similar to (i), with identity element $\mathbf{1}$ for g. □

Theorem 4.2.8. If $\cong$ is a congruence relation and $a_i \cong b_i, c_j \cong d_j, x_t \cong y_t, z_k \cong u_k$ $\forall a_i, b_i, c_j, d_j, x_t, y_t, z_k, u_k \in \mathcal{U}$ where $1 \leq i \leq m, 1 \leq j \leq n$ then

(i) $f(f(\overset{(t)}{a_1^m}), x_{t+1}^m) \cong f(f(\overset{(t)}{b_1^m}), y_{t+1}^m)$

for all $1 \leq t \leq m$

(ii) $f(g(\overset{(t)}{c_1^n}), x_{t+1}^m) \cong f(g(\overset{(t)}{d_1^n}), y_{t+1}^m)$

for all $1 \leq t \leq m$

(iii) $g(f(\overset{(k)}{a_1^m}), z_{k+1}^n) \cong g(f(\overset{(k)}{b_1^m}), u_{k+1}^n)$

for all $1 \leq k \leq n$

(iv) $g(g(\overset{(k)}{c_1^n}), z_{k+1}^n) \cong g(g(\overset{(k)}{d_1^n}), u_{k+1}^n)$

for all $1 \leq k \leq n$

Proof. (i) From Lemma 4.2.7, if $a_i \cong b_i$, $\forall a_i, b_i \in \mathcal{U}$ and $1 \leq i \leq m$ then

$$f(f(\overset{(t)}{a_1^m}), \underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-t}) \cong f(f(\overset{(t)}{b_1^m}), \underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-t}) \tag{4.9}$$

for all $1 \leq t \leq m$.

From Definition 4.2.1, if $x_t \cong y_t$, for all $1 \leq t \leq m$ then

$$x_{t+1} \cong y_{t+1} \tag{4.10}$$

by f operation on both sides of (4.9) with y_{t+1}, we get

$$\begin{aligned} & f(f(f(\overset{(t)}{a_1^m}), \underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-t}), y_{t+1}, \underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-2}) \\ \cong\ & f(f(f(\overset{(t)}{b_1^m}), \underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-t}), y_{t+1}, \underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-2}) \end{aligned},$$

$$f(f(\overset{(t)}{a_1^m}), y_{t+1}, \underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-t-1}) \cong f(f(\overset{(t)}{b_1^m}), y_{t+1}, \underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-t-1}) \tag{4.11}$$

by f operation on both sides of (4.10) with $f(f(\overset{(t)}{a_1^m}), \underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-t})$, we get

$$\begin{aligned} &f(f(f(\overset{(t)}{a_1^m}), \underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-t}), x_{t+1}, \underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-2}) \\ \cong\ &f(f(f(\overset{(t)}{a_1^m}), \underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-t}), y_{t+1}, \underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-2}) \end{aligned}$$

By Law of Associativity we get the following result

$$f(f(\overset{(t)}{a_1^m}), x_{t+1}, \underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-t-1}) \cong f(f(\overset{(t)}{a_1^m}), y_{t+1}, \underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-t-1}) \tag{4.12}$$

From (4.11) and (4.12), we get the following

$$f(f(\overset{(t)}{a_1^m}), x_{t+1}, \underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-t-1}) \cong f(f(\overset{(t)}{b_1^m}), y_{t+1}, \underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-t-1}).$$

Similarly, we can get the following

$$f(f(\overset{(t)}{a_1^m}), x_{t+1}^m) \cong f(f(\overset{(t)}{b_1^m}), y_{t+1}^m)$$

In a similar manner we can prove (ii) with identity element **0** and (iii) and (iv) with identity element **1**. $\square$

Theorem 4.2.9. If $\cong$ is a congruence relation and $a_i \cong b_i, c_j \cong d_j, x_i \cong y_i, z_j \cong u_j$ for all $a_i, b_i, c_j, d_j, x_i, y_i, z_j, u_j \in \mathcal{U}$ for all $1 \leq i \leq m, 1 \leq j \leq n$ and $t < m, k < n$, where $t, k \in \mathbb{Z}_+$ then

(i) $f(\overset{(t)}{f}(a_1^m), \overset{(m-t)}{f}(x_1^m)) \cong f(\overset{(t)}{f}(b_1^m), \overset{(m-t)}{f}(y_1^m))$

for all $1 \leq t \leq m$,

(ii) $f(\overset{(t)}{g}(c_1^n), \overset{(m-t)}{g}(z_1^n)) \cong f(\overset{(t)}{g}(d_1^n), \overset{(m-t)}{g}(u_1^n))$

for all $1 \leq t \leq m$,

(iii) $g(\overset{(k)}{f}(a_1^m), \overset{(n-k)}{f}(x_1^m)) \cong g(\overset{(k)}{f}(b_1^m), \overset{(n-k)}{f}(y_1^m))$

for all $1 \leq k \leq n$,

(iv) $g(\overset{(k)}{g}(c_1^n), \overset{(n-k)}{g}(z_1^n)) \cong g(\overset{(k)}{g}(d_1^n), \overset{(n-k)}{g}(u_1^n))$

for all $1 \leq k \leq n$.

Proof. (i) From Lemma 4.2.7, if $a_i \cong b_i \ \forall a_i, b_i \in \mathcal{U}$ then

$$f(\overset{(t)}{f}(a_1^m), \underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-t}) \cong f(\overset{(t)}{f}(b_1^m), \underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-t}) \tag{4.13}$$

for all $1 \leq i \leq m$ and $1 \leq t \leq m$.

From Definition 4.2.1, if $x_i \cong y_i$, for all $1 \leq i \leq m$ then

$$f(x_1^m) \cong f(y_1^m) \tag{4.14}$$

by f operation on both sides of (4.13) with $f(y_1^m)$, we get

$$f(f(f(\overset{(t)}{a_1^m}),\underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-t}),f(y_1^m),\underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-2})$$
$$\cong f(f(f(\overset{(t)}{b_1^m}),\underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-t}),f(y_1^m),\underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-2}).$$

Which can be deduced as follows

$$f(f(\overset{(t)}{a_1^m}),f(y_1^m),\underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-t-1})\cong f(f(\overset{(t)}{b_1^m}),f(y_1^m),\underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-t-1}) \tag{4.15}$$

by f operation on both sides of (4.14) with $f(f(\overset{(t)}{a_1^m}),\underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-t})$, we get

$$f(f(x_1^m),f(f(\overset{(t)}{a_1^m}),\underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-t}),\underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-2})$$
$$\cong f(f(y_1^m),f(f(\overset{(t)}{a_1^m}),\underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-t}),\underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-2}).$$

From above expression, we get the following result

$$f(f(x_1^m),f(\overset{(t)}{a_1^m}),\underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-t-1})\cong f(f(y_1^m),f(\overset{(t)}{a_1^m}),\underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-t-1}). \tag{4.16}$$

From (4.15) and (4.16), we get the following

$$f(f(\overset{(t)}{a_1^m}),f(x_1^m),\underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-t-1})\cong f(f(\overset{(t)}{b_1^m}),f(y_1^m),\underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-t-1}).$$

Similarly, we can get the following

$$f(f(\overset{(t)}{a_1^m}), \overset{(m-t)}{f(x_1^m)}) \cong f(f(\overset{(t)}{b_1^m}), \overset{(m-t)}{f(y_1^m)}).$$

Similarly, we can prove for (ii) with identity element **0** and (iii) and (iv) with identity element **1**. $\square$

Bibliography

[1] Abbott, R. J. Resourceful systems for fault tolerance, reliability, and safety. *ACM Comput. Surv. 22*, 1 (1990), 35–68.

[2] Allen, P. J. A fundamental theorem of homomorphisms for semirings. *Proceedings of the American Mathematical Society 21*, 2 (1969), 412–416.

[3] Annamalai, V., Gupta, S. K. S., and Schwiebert, L. On tree-based convergecasting in wireless sensor networks. In *in Proceedings of the IEEE Wilress Communications and Networking Conference (WCNC03* (2003).

[4] Avižienis, A. Design of fault-tolerant computers. In *AFIPS '67 (Fall): Proceedings of the November 14-16, 1967, fall joint computer conference* (New York, NY, USA, 1967), ACM, pp. 733–743.

[5] Avižienis, A. The dependability problem: Introduction and verification of fault tolerance for a very complex system. In *ACM '87: Proceedings of the 1987 Fall Joint Computer Conference on Exploring technology: today and tomorrow* (Los Alamitos, CA, USA, 1987), IEEE Computer Society Press, pp. 89–93.

[6] Ayav, T., Fradet, P., and Girault, A. Implementing fault-tolerance in real-time programs by automatic program transformations. *ACM Trans. Embed. Comput. Syst. 7*, 4 (2008), 1–43.

[7] Beckmann, P. E. *Fault-Tolerant Computation using Algebraic Homomorphisms.* Phd thesis, Department of Electrical Engineering and Computer Science, Massachusetts Institute of Technology, Cambridge, Massachusetts, 1992.

[8] Bourne, S. On multiplicative idempotents of a potent semiring. *Proceedings of the National Academy of Sciences of the United States of America 42*, 9 (1956), 632–638.

[9] Brière, D., and Traverse, P. AIRBUS A320/A330/A340 electrical flight controls: A family of fault-tolerant systems. In *FTCS* (1993), pp. 616–623.

[10] Burris, S., and Sankappanavar, H. P. *A Course in Universal Algebra.* No. 78 in Graduate Texts in Mathematics. Springer-Verlag, 1981.

[11] Cristian, F. Understanding fault-tolerant distributed systems. *Commun. ACM 34*, 2 (1991), 56–78.

[12] Cristian, F., Dancey, B., and Dehn, J. Fault-tolerance in the advanced automation system. In *EW 4: Proceedings of the 4th workshop on ACM SIGOPS European workshop* (New York, NY, USA, 1990), ACM, pp. 6–17.

[13] Cristian, F., Dancey, B., and Dehn, J. Fault-tolerance in air traffic control systems. *ACM Trans. Comput. Syst. 14*, 3 (1996), 265–286.

[14] Davvaz, B., and Dudek, W. A. Fuzzy n-ary groups as a generalization of rosenfield's fuzzy groups. *Journal of Multiple-Valued Logic and Soft Computing 15*, 5-6 (2009), 471–488.

[15] Davvaz, B., Dudek, W. A., and Mirvakili, S. Neutral elements, fundamental relations and n-ary hypersemigroups. *International Journal of Algebra and Computation 19*, 4 (2009), 567–583.

[16] Davvaz, B., Dudek, W. A., and Vougiouklis, T. A generalization of n-ary algebraic systems. *Communications in Algebra 37* (2009), 1248–1263.

[17] De Capua, C., Battaglia, A., Meduri, A., and Morello, R. A patient-adaptive ECG measurement system for fault-tolerant diagnoses of heart abnormalities. In *Instrumentation and Measurement Technology Conference Proceedings, 2007. IMTC 2007. IEEE* (may 2007), pp. 1–5.

[18] Dudek, W. A. Remarks on n-groups. *Demonstratio Math.13* (1980), 165–181.

[19] Dudek, W. A. On distributive n-ary groups. *Quasigroups and Related Systems* (1995), 132–151.

[20] Dudek, W. A. Idempotents in n-ary semigroups. *Southeast Asian Bulletin of Mathematics 25*, 1 (2001), 97–104.

[21] Dudek, W. A., and Mukhin, V. V. On topological n-ary semigroups. *Quasigroups and related systems*, 3 (1996), 73–88.

[22] Ferrell, C. Failure recognition and fault tolerance of an autonomous robot. *Adaptive Behavior 2* (1994), 375–398.

[23] Ghosh, S. A characterization of semirings which are subdirect products of a distributive lattice and a ring. *Semigroup Forum 59*, 1 (1999), 106–120.

[24] Gluskin, L. M. Fault tolerant planning for critical robots. *Mat.Sbornik 68* (1965), 444–472.

[25] Golan, J. S. *Semirings and Their Applications*. Kluwer Academic Publishers, 1999.

[26] Green, T. J., Karvounarakis, G., and Tannen, V. Provenance semirings. In *PODS '07: Proceedings of the twenty-sixth ACM SIGMOD-SIGACT-SIGART symposium on Principles of database systems* (New York, NY, USA, 2007), ACM, pp. 31–40.

[27] Hadjicostis, C. N. Fault-tolerant computation in semigroups and semirings, mit m.eng. thesis, 1995.

[28] Hebisch, U., and Weinert, H. J. *Semirings: Algebraic Theory and Applications In Computer Science*. World Scientific, Singapore, 1998.

[29] Hooi-Tong, L. Notes on semirings. *Mathematics Magazine 40*, 3 (1967), 150–152.

[30] Jalote, P. *Fault Tolerance in Distributed Systems*. Prentice-Hall, Inc., 1994.

[31] Koushanfar, F., Potkonjak, M., and Sangiovanni-vincentelli, A. Fault tolerance in wireless sensor networks, book chapter. In *in Handbook of Sensor Networks, I. Mahgoub and M. Ilyas*, p. 2004.

[32] Leeson, J. J., and Butson, A. T. On the general theory of (m, n) rings. *Algebra Universalis 11*, 1 (1980), 42–76.

[33] Lussier, B., Gallien, M., Guiochet, J., Ingrand, F., Killijian, M.-O., and Powell, D. Fault tolerant planning for critical robots. *Dependable Systems and Networks, International Conference on 0* (2007), 144–153.

[34] Monico, C. J. *Semirings and Semigroup Actions in Public-Key Cryptography*. Phd thesis, Department of Mathematics, Graduate School of the University of Notre Dame, Notre Dame, Indiana, 2002.

[35] Perraju, T., Rana, S., and Sarkar, S. Specifying fault tolerance in mission critical systems. In *High-Assurance Systems Engineering Workshop, 1996. Proceedings., IEEE* (oct 1996), pp. 24–31.

[36] Rao, S. *Safety And Hazard Analysis In Concurrent Systems*. Phd thesis, The University of Iowa, 2005.

[37] Rao, S. An algebra of fault tolerance. *Journal of Algebra and Discrete Structures 6*, 3 (2008), 161–180.

[38] Rao, S. A systems algebra and its applications. In *Systems Conference, 2008 2nd Annual IEEE* (2008), pp. 1–8.

[39] Timm, J. Kommutative n-gruppen. *Dissertation* (1967).